Chapter 4. — Linear superpositions

CHAPTER 4. — LINEAR SUPERPOSITIONS

In this chapter we prove that there exist analytic functions which are not representable by means of linear superpositions of smooth functions of one variable.

§ 1. *Notation*

Throughout we assume that all the functions are defined and continuous for all values of the arguments. If we say that a function is continuously differentiable, we mean by this that its first partial derivatives are defined and continuous for all values of the arguments; $z = (x, y)$ is the point of the plane with coordinates x and y; grad $[q(z)]$ is the gradient of the function $q(z)$, that is, the vector-function with coordinates $\frac{\partial q}{\partial x}$ and $\frac{\partial q}{\partial y}$;

$$D\left(\frac{q_1, q_2}{x, y}\right) = \begin{vmatrix} \frac{\partial q_1}{\partial x} & \frac{\partial q_1}{\partial y} \\ \frac{\partial q_2}{\partial x} & \frac{\partial q_2}{\partial y} \end{vmatrix}$$

is the Jacobian of the pair of functions q_1 and q_2.

$q(D)$ is the image of the set D under the mapping effected by the function $q(x, y)$; $q^{-1}(\delta)$ is the complete inverse image of the interval δ on the axis of values of the function $q(x, y)$.

$e(q, t)$ is the set of level t of the function $q = q(x, y)$.

$\tau(e, z)$ is the unit tangent vector to the curve e at the point $z \in e$.

$\gamma(\tau_1, \tau_2)$ is the absolute value of the acute angle between the vectors τ_1 and τ_2.

$h_1(e)$ is the length of the set e.

$d_1(e)$ is the one-dimensional diameter of the set e.

$0(\gamma)$ is a quantity bounded by a constant depending only on γ.

$\rho(A_1, A_2)$ is the distance between the sets A_1 and A_2 in the sense of deviation, more precisely

$$\rho(A_1, A_2) = \max\{\sup_{z_1 \in A_1} \inf_{z_2 \in A_2} \rho(z_1, z_2), \sup_{z_2 \in A_2} \inf_{z_1 \in A_1} \rho(z_1, z_2)\},$$

where $\rho(z_1, z_2)$ is the distance between the points z_1 and z_2.

§ 2. *Estimate of the difference of the integrals of one term of a superposition along nearby level curves*

Let G be a region of the plane of the variables x and y, and $q_1(x, y)$ and $q_2(x, y)$ continuously differentiable functions satisfying in this region the following conditions: a) the partial derivatives with respect to x and with respect to y have modulus of continuity $\omega(\delta)$; b) the inequalities

$$0 < \gamma \leqslant |\text{ grad } [q_i(x, y)]| \leqslant \frac{1}{\gamma} < \infty \qquad (i = 1, 2)$$

are satisfied everywhere in G, where γ is a constant; c) for any point $(x, y) \in G$ the absolute value of the acute angle formed by the level curves of the functions $q_1(x, y)$ and $q_2(x, y)$ which pass through this point is greater than some positive constant γ.

LEMMA 4.2.1. *Let e'_{q2} and e''_{q2} be two level curves of the function q_2 and e'_{q1} and e''_{q1} level curves of the function q_1; $[a', a''] \subset G$ the segment of the curve e'_{q1} with end-points $a' \in e'_{q2}$ and $a'' \in e''_{q2}$; $[b', b'']$ the segment of the curve e''_{q1} with end-points $b' \in e'_{q2}$ and $b'' \in e''_{q2}$. Then*

$$h_1([b', b'']) \leqslant h_1([a', a'']) \times (1 + c_1(\gamma)\, \omega(\delta)),$$

where $\delta = d_1([a', a''] \cup [b', b''])$ and $c_1(\gamma)$ depends only on γ.

Proof. Since $q_2(a'') - q_2(a') = q_2(b'') - q_2(b')$, we have

$$\int\limits_{s \in [a', a'']} \frac{\partial q_2}{\partial s}\, ds = \int\limits_{s \in [b', b'']} \frac{\partial q_2}{\partial s}\, ds .$$

Consequently, $\dfrac{\partial q_2(a^*)}{\partial s} h_1([a', a'']) = \dfrac{\partial q_2(b^*)}{\partial s} h_1([b', b''])$, where $\dfrac{\partial q_2(a^*)}{\partial s}$ and $\dfrac{\partial q_2(b^*)}{\partial s}$ are the derivatives at the points $a^* \in [a', a'']$ and $b^* \in [b', b'']$ along the curves $[a', a'']$ and $[b', b'']$, respectively. We show that $\dfrac{\partial q_2(a^*)}{\partial s} = \dfrac{\partial q_2(b^*)}{\partial s} + O(\gamma)\, \omega(\delta)$. We denote by q_2^* the derivative of q_2 at the point b^* in the direction of $\tau(e'_{q1}, a^*)$ and put $\alpha = \gamma\{\tau[e''_{q1}, b^*], \tau[e'_{q1}, a^*]\}$. From conditions a) and b) it follows that $\dfrac{\partial q_2(a^*)}{\partial s} = q_2^* + O(1)\, \omega(\delta)$ and α

$= O(\gamma)\,\omega(\delta)$. We denote by β_1 and β_2 the values of the angles formed by the vectors $\tau\,[e''_{q_1}, b^*]$ and $\tau\,[e'_{q_1}, a^*]$ with the vector grad $[q_2(b^*)]$. We have

$$\left| q_2^* - \frac{\partial q_2(b^*)}{\partial s} \right| = |\,\mathrm{grad}\,[q_2(b^*)]\,|\ \ |\cos\beta_2 - \cos\beta_1| = O(\gamma)\,\alpha$$
$$= O(\gamma)\,\omega(\delta)\,.$$

Thus,

$$\frac{\partial q_2(a^*)}{\partial s} = q_2^* + O(1)\,\omega(\delta) = \frac{\partial q_2(b^*)}{\partial s}$$

$$+ O(1)\left\{ \left| q_2^* - \frac{\partial q_2(b^*)}{\partial s} \right| + \omega(\delta) \right\} = \frac{\partial q_2(b^*)}{\partial s} + O(\gamma)\,\omega(\delta)\,.$$

Consequently,

$$h_1([b', b'']) = h_1([a', a''])\,\frac{\partial q_2(a^*)}{\partial s}\left(\frac{\partial q_2(b^*)}{\partial s}\right)^{-1}$$
$$= h_1([a', a''])\left(1 + O(\gamma)\,\omega(\delta)\left(\frac{\partial q_2(b^*)}{\partial s}\right)^{-1}\right)$$
$$= h_1([a', a''])\,(1 + O(\gamma)\,\omega(\gamma))\,,$$

since by virtue of b) $\dfrac{\partial q_2(b^*)}{\partial s} > |\,\mathrm{grad}\,[q_2(b^*)]\,|\sin\gamma$. This, proves the lemma.

Lemma 4.2.2. *Let* $q_m(x, y)$ $(m = 1, 2, \ldots, N)$ *be continuously differentiable functions. In any region* D *we can find a subregion* $G \subset D$, *determine a constant* $\gamma > 0$, *and renumber the functions* $\{q_m(x, y)\}$ *with two indices so that the functions*

$$q_i^k(x, y) = q_m(x, y) \quad (i = 0, 1, 2, \ldots, n\,;\ k = 1, 2, \ldots, m_i\,;\ \sum_{i=0}^{n} m_i = N)$$

obtained after the renumbering satisfy the following conditions:

(1) *when* $i = 0$, $q_i^k = \mathrm{const}$ *in* G, *and when* $i > 0$, $\gamma \leqslant |\,\mathrm{grad}\,[q_i^k(x, y)]\,| \leqslant \dfrac{1}{\gamma}$ *for every point* $(x, y) \in G$;

(2) *the functions* $q_i^k(x, y)$ $(i > 0$ *fixed,* $k = 1, 2, \ldots, m_i)$ *have in the region* G *identical sets of level curves, more precisely, in the region* G, $q_i^k(x, y) \equiv \varphi_i^{k,l}(q_i^l(x, y))$, *where* $\varphi_i^{k,l}(t)$ *is a strictly monotonic continuously differentiable function of* t;

(3) *when* $i \neq j\,(i, j \neq 0)$, *then for any* k *and* l *the absolute value of the acute angle formed by the level curves of the functions* $q_i^k(x, y)$ *and* $q_j^l(x, y)$ *which pass through an arbitrary point* $(x, y) \in G$ *is greater than* γ.

Proof. By the continuity of the partial derivatives of the functions $\{ q_m(x, y) \}$ there exists a subregion $G^* \subset D$ inside which for any function $q_m(x, y)$ either $\operatorname{grad} q_m(x, y) \equiv 0$ or $|\operatorname{grad} q_m(x, y)|$ is greater than some positive constant. From the continuity of the partial derivatives of the functions $\{ q_m(x, y) \}$ it follows also that there exists a subregion $G^{**} \subset G^*$ inside which for any pair of functions $q_r(x, y)$ and $q_s(x, y)$ one of two conditions holds: either $D\left(\dfrac{q_r, q_s}{x, y}\right) \equiv 0$ in G^{**}, or for every point of G^{**} the level curves of $q_r(x, y)$ and $q_s(x, y)$ that pass through this point intersect at a non-zero angle $\left(D\left(\dfrac{q_r, q_s}{x, y}\right) \neq 0 \text{ in } G^{**}\right)$. From the implicit function theorem it follows that there exists a subregion $G \subset G^{**}$ in which condition (2) is satisfied for every pair of functions $q_r(x, y)$ and $q_s(x, y)$ with gradients different from zero and with determinant $D\left(\dfrac{q_r, q_s}{x, y}\right) \equiv 0$.

We now renumber the functions $\{ q_m(x, y) \}$ with two indices in such a way that only functions constant in G have lower index zero, and the same lower index is assigned to those functions whose level curves coincide identically in G. This proves the lemma.

We consider in the region G a superposition of the form $\sum\limits_{i=0}^{n} \sum\limits_{k=1}^{m_i} p_r(x, y)\, f_i^k(q_i^k(x, y))$, where $\{ f_i^k(t) \}$ are continuous functions of one variable, $\{ p_i^k(x, y) \}$ are continuous functions satisfying in G the condition $|p_i^k(x, y)| \leqslant \dfrac{1}{\gamma}$ and $\{ q_i^k(x, y) \}$ are continuously differentiable functions satisfying in G conditions (1), (2), (3) of Lemma 4.2.2. Let $\omega(\delta)$ be the common modulus of continuity in G of the functions $\left\{ p_i^k(x, y)\, ; \dfrac{\partial q_i^k(x, y)}{\partial x}\, ; \dfrac{\partial q_i^k(x, y)}{\partial y} \right\}$. Let $[a', a'']$ and $[b', b'']$ be segments of the level curves of the functions $\{ q_i^k(x, y) \}$ ($i > 0$ fixed) lying in G. Let

$$\alpha = h_1([a', a''])\, ; \quad \delta = \rho([a', a''],\ [b', b''])\, ;$$

$$\varepsilon = \sup \left| \sum_{i=0}^{n} \sum_{k=1}^{m_i} p_i^k(x, y) f_i^k(q_i^k(x, y)) \right| ;$$

$$m = \max_{i,k} \sup \left| f_i^k (q_i^k (x, y)) \right| ,$$

where sup is taken over all points $(x, y) \in [a', a''] \cup [b', b'']$.

LEMMA 4.2.3. *If δ is sufficiently small $(\omega(\delta) \leqslant C_2(\gamma))$, then for any $i > 0$*

$$\left| \int\limits_{s \in [a', a'']} \sum_{k=1}^{m_i} p_i^k (s) f_i^k (q_i^k (s))\, ds - \int\limits_{s \in [b', b'']} \sum_{k=1}^{m_i} p_i^k (s) f_i^k (q_i^k (s))\, ds \right|$$
$$\leqslant C_3(\gamma)(\alpha\varepsilon + m\alpha\omega(\delta) + m\delta),$$

where the constants $C_2(\gamma)$, $C_3(\gamma)$ depend only on γ.

Proof. By (1), (2), (3) there exists a sufficiently small constant $C_2(\gamma)$ and a sufficiently large constant $C_3(\gamma)$ such that if $\omega(\delta) \leqslant C_2(\gamma)$ and for a point $a \in [a', a'']$ the inequalities $h_1([a', a]) \geqslant C_3(\gamma)\, \delta$; $h_1([a, a'']) \geqslant C_3(\gamma)\delta$ are satisfied, then for any $j \neq i$ $(j > 0)$ the level curve of the function q_j^k that passes through a intersects $[b', b'']$ of the level curve of q_i^k. Suppose that $\alpha > 2C_3(\gamma)\,\delta$ (if $\alpha \leqslant 2\, C_3(\gamma)\, \delta$, then the assertion of the lemma is trivial) and suppose that the segment $[\tilde{a}', \tilde{a}'']$ of the level curve of q_i^k is such that $[\tilde{a}', \tilde{a}''] \subset [a', a'']$ and $h_1([a', \tilde{a}']) = h_1([\tilde{a}'', a'']) = C_3(\gamma)\, \delta$. On the arc $[\tilde{a}', \tilde{a}'']$ we fix a system of points $a_1, a_2, \ldots, a_\nu$ $(\tilde{a}' = a_1,\ \tilde{a}'' = a_\nu)$, uniformly distributed along the length of this arc, and denote by b_r the point of intersection of $[b', b'']$ with the level curve of q_j^k that passes through a_r (here $j \neq i$ should for the time being be regarded as fixed). Using Lemma 4.2.1 we have

$$\left| \int\limits_{s \in [a', a'']} p_j^k (s) f_j^k (q_j^k (s))\, ds - \int\limits_{s \in [b', b'']} p_j^k (s) f_j^k (q_j^k (s))\, ds \right|$$
$$= \left| \int\limits_{s \in [a_1, a_\nu]} p_j^k (s) f_j^k (q_j^k (s))\, ds - \int\limits_{s \in [b_1, b_\nu]} p_j^k (s) f_j^k (q_j^k (s))\, ds \right|$$
$$+ O(\gamma)\, m\delta$$
$$= \lim_{\nu \to \infty} \Big| \sum_{r=1}^{\nu} p_j^k (a_r) f_j^k (q_j^k (a_r))\, h_1([a_r, a_{r+1}])$$
$$- \sum_{r=1}^{\nu} p_j^k (b_r) f_j^k (q_j^k (b_r))\, h_1([b_r, b_{r+1}]) \Big| + O(\gamma)\, m\delta$$

$$= \lim_{\nu\to\infty} \Big| \sum_{r=1}^{\nu} p_j^k(a_r) f_j^k(q_j^k(a_r))\, h_1([a_r, a_{r+1}])$$

$$- \sum_{r=1}^{\nu} p_j^k(a_r) f_j^k(q_j^k(a_r))\, h_1([a_r, a_{r+1}]) (1 + O(\gamma)\,\omega(\delta))$$

$$+ \sum_{r=1}^{\nu} (p_j^k(a_r) - p_j^k(b_r)) f_j^k(q_j^k(a_r))\, h_1([b_r, b_{r+1}]) \Big| + O(\gamma)\, m\delta$$

$$= \lim_{\nu\to\infty} \Big| \sum_{r=1}^{\nu} p_j^k(a_r) f_j^k(q_j^k(a_r))\, h_1([a_r, a_{r+1}])\, O(\gamma)\,\omega(\delta)$$

$$+ \sum_{r=1}^{\nu} f_j^k(q_j^k(a_r))\, h_1([b_r, b_{r+1}])\, O(\gamma)\,\omega(\delta) \Big| + O(\gamma)\, m\delta$$

$$= O(\gamma)\, m\alpha\omega(\delta) + O(\gamma)\, m\alpha\omega(\delta) + O(\gamma)\, m\delta = O(\gamma)\, m(\delta + \alpha\omega(\delta)).$$

Then

$$\Big| \int\limits_{s\in[a', a'']} \sum_{k=1}^{m_i} p_i^k(s) f_i^k(q_i^k(s))\, ds - \int\limits_{s\in[b', b'']} \sum_{k=1}^{m_i} p_i^k(s) f_i^k(q_i^k(s))\, ds$$

$$\leqslant \Big| \int\limits_{s\in[a', a'']} \sum_{i=0}^{n} \sum_{k=1}^{m_i} p_i^k(s) f_i^k(q_i^k(s))\, ds - \int\limits_{s\in[b', b'']} \sum_{i=0}^{n} \sum_{k=1}^{m_i} p_i^k(s) f_i^k(q_i^k(s))\, ds \Big|$$

$$+ \sum_{j\neq i} \Big| \int\limits_{s\in[a', a'']} \sum_{k=1}^{m_j} p_j^k(s) f_j^k(q_j^k(s))\, ds - \int\limits_{s\in[b', b'']} \sum_{k=1}^{m_j} p_j^k(s) f_j^k(q_j^k(s))\, ds \Big|$$

$$\leqslant C_4(\gamma)\,\alpha\varepsilon + n(\max_{j\neq i} m_j)\, C_5(\gamma)\, m(\delta + \alpha\omega(\delta))$$

$$\leqslant C_3(\gamma)\,(\alpha\varepsilon + m\delta + m\alpha\omega(\delta)).$$

This proves the lemma.

§ 3. *Deletion of dependent terms*

On a bounded closed set D we consider the space of linear superpositions of the form $\sum_{k=1}^{m} p_k(x, y) f_k(q(x, y))$, $(x, y) \in D$. Here the functions $\{p_k(x, y)\}$ and $q(x, y)$ are continuous and fixed, and $\{f_k(t)\}$ are arbitrary continuous functions of one variable. We assume that the function $q(x, y)$ is such that for any sequence $t_n \in q(D) \to t \in q(D)$ we have $\rho\, [e(q, t_n) \cap D, e(q, t) \cap D] \to 0$. We put

$$\lambda(t, D, q, p_1, \dots, p_m) = \inf_{\{c_k\}} \sup_{(x, y)\in e(q, t)\cap D} \Big| \sum_{k=1}^{m} c_k p_k(x, y) \Big|,$$

where inf is taken over all sets of numbers $\{ c_k \}$ for which $\max_k | c_k | = 1$. The function $\lambda(t, D, q, \{ p_k \})$, as a function of t, is defined only on the set $q(D)$.

LEMMA 4.3.1. *The function* $\lambda(t, D, q, \{ p_k \})$ *depends continuously on* t.

Proof. The linear combinations $\sum_{k=1}^{m} c_k p_k(x, y)$ for all possible systems of numbers $\{ c_k \}$ for which $\max_k | c_k | \leqslant 1$, form an equicontinuous set of functions, considered on the bounded closed set D. Consequently, for any $\varepsilon > 0$ there is a $\delta > 0$ such that if $| t_1 - t_2 | < \delta$, then

$$\left| \sup_{(x, y) \in e(q, t_1)} \left| \sum_{k=1}^{m} c_k p_k(x, y) \right| - \sup_{(x, y) \in e(q, t_2)} \left| \sum_{k=1}^{m} c_k p_k(x, y) \right| \right| < \varepsilon$$

simultaneously for all systems of numbers $\{ c_k \}$ such that $\max_k | c_k | \leqslant 1$. For definiteness, suppose that $\lambda(t_2, D, q, \{ p_k \}) \geqslant \lambda(t_1, D, q, \{ p_k \})$. Since the expression $\sup_{(x, y) \in e(q, t_1)} \left| \sum_{k=1}^{m} c_k p_k(x, y) \right|$ depends continuously on the coefficients $\{ c_k \}$, there exists a system of numbers $\{ c_k^1 \}$ such that $\max_k | c_k^1 | = 1$ and

$$\lambda(t_1, D, q, \{ p_k \}) = \sup_{(x, y) \in e(q, t_1)} \left| \sum_{k=1}^{m} c_k^1 p_k(x, y) \right| .$$

Since

$$\lambda(t_2, D, q, \{ p_k \}) \leqslant \sup_{(x, y) \in e(q, t_2)} \left| \sum_{k=1}^{m} c_k^1 p_k(x, y) \right| ,$$

we have

$$0 \leqslant \lambda(t_2) - \lambda(t_1) \leqslant \sup_{(x, y) \in e(q, t_2)} \left| \sum_{k=1}^{m} c_k^1 p_k(x, y) \right| - \sup_{(x, y) \in e(q, t_1)} \left| \sum_{k=1}^{m} c_k^1 p_k(x, y) \right| < \varepsilon .$$

This proves the lemma.

LEMMA 4.3.2. *The function* $\lambda(t, D, q, \{ p_k \})$ *depends continuously on* D *in the sense that there exists a function* $\mu(\varepsilon) \to 0$ *as* $\varepsilon \to 0$, *having the property: if the set* $D_\varepsilon \subset D$ *is such that, for any* t, $D_\varepsilon \cap e(q, t)$ *forms an* ε*-net in the set* $e(q, t) \cap D$, *then*

$$\max_{t \in q(D)} \left| \lambda(t, D, q, \{p_k\}) - \lambda(t, D_\varepsilon, q, \{p_k\}) \right| \leqslant \mu(\varepsilon) .$$

Proof. Using the equicontinuity of the set of functions $\sum_{k=1}^{n} c_k p_k(x, y)$ where $\max_k |c_k| \leqslant 1$, we conclude that there exists a function $\mu(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that the inequality

$$0 \leqslant \sup_{(x,y) \in e(q,t) \cap D} \left| \sum_{k=1}^{m} c_k p_k(x, y) \right| - \sup_{(x,y) \in e(q,t) \cap D_\varepsilon} \left| \sum_{k=1}^{m} c_k p_k(x, y) \right| \leqslant \mu(\varepsilon).$$

uniformly over all $t \in q(D)$ and over all systems of numbers $\{c_k\}$ for which $\max_k |c_k| \leqslant 1$. For any $\varepsilon > 0$ there exists a system of numbers $\{c_k^\varepsilon\}$ such that $\max_k |c_k^\varepsilon| = 1$ and

$$\lambda(t, D_\varepsilon, q, \{p_k\}) = \sup_{(x,y) \in e(q,t) \cap D_\varepsilon} \left| \sum_{k=1}^{m} c_k^\varepsilon p_k(x, y) \right| .$$

Since for any ε

$$\lambda(t, D, q, \{p_k\}) \leqslant \sup_{(x,y) \in e(q,t) \cap D} \left| \sum_{k=1}^{m} c_k^\varepsilon p_k(x, y) \right|$$

and, on the other hand, $\lambda(t, D, q, \{p_k\}) \geqslant \lambda(t, D_\varepsilon, q, \{p_k\})$ (we recall that $D_\varepsilon \subset D$), we have

$$0 \leqslant \lambda(t, D, q, \{p_k\}) - \lambda(t, D_\varepsilon, q, \{p_k\}) \leqslant \sup_{(x,y) \in e(q,t) \cap D} \left| \sum_{k=1}^{m} c_k^\varepsilon p_k(x, y) \right| - \sup_{(x,y) \in e(q,t) \cap D_\varepsilon} \left| \sum_{k=1}^{m} c_k^\varepsilon p_k(x, y) \right| < \mu(\varepsilon) .$$

This proves the lemma.

LEMMA 4.3.3. *Let F be a closed set on the t-axis; $F \subset q(D)$. For every $t \in F$, suppose that there exists one and only one system of numbers $\{C_k\}$ ($\max_k |C_k| = 1$) such that* $\sum_{k=1}^{m} C_k p_k(x, y) \equiv 0$ on the set $e(q, t) \cap D$. *Then each of the functions $\{C_k(t)\}$ depends continuously on t on the set F.*

Proof. Suppose that $t_n \in F$, $t \in F$ and $t_n \to t$. We put $\overline{\lim_{n\to\infty}} C_k(t_n) = \tilde{C}_k$ and $\underline{\lim_{n\to\infty}} C_k(t_n) = \tilde{\tilde{C}}_k$. Since $\sum_{k=1}^{m} C_k(t_n) p_k(x, y) \equiv 0$ on the set $e(q, t_n) \cap D$ and $\rho[e(q, t) \cap D, e(q, t_n) \cap D] \to 0$ as $n \to \infty$, we have $\sum_{k=1}^{m} \tilde{C}_k p_k(x, y)$

$\equiv 0 \equiv \sum_{k=1}^{m} \tilde{\tilde{C}}_k\, p_k(x, y)$ on the set $e(q, t) \cap D$. Consequently, by the condition of the lemma, $\tilde{C}_k = \tilde{\tilde{C}}_k = C_k(t)$. This proves the lemma.

LEMMA 4.3.4. *Suppose that* $\lambda(t, D, q, \{p_k\}) \equiv 0$ *on some non-empty portion* δ *of the set* $q(D)$. *Then there is a non-empty portion* $\delta^* \subset \delta$ *and an index* l *such that for any continuous functions* $\{f_k(t)\}$ *there are continuous functions* $\{f_k^*(t)\}$ *such that*

$$\sum_{k \neq l} f_k^*(q(x, y))\, p_k(x, y) = \sum_{k=1}^{m} f_k(q(x, y))\, p_k(x, y)$$

on the set $q^{-1}(\delta^*) \cap D$.

We recall that a portion δ of a set E is that part of it which lies in the interval δ.

Proof. We prove the lemma by induction on m. For $m = 1$ the assertion of the lemma is obvious. We denote by δ_k the set of all points t of the portion δ for which $\lambda(t, D, q, p_1, \ldots, p_{k-1}, p_{k+1}, \ldots, p_m) = 0$. By Lemma 4.3.1, the set is closed. Two cases are possible.

1) For some k the set δ_k contains a non-empty portion δ_k' of the set $q(D)$. Since $\lambda(t, D, q, p_1, \ldots, p_{k-1}, p_{k+1}, \ldots, p_m) = 0$ for every $t \in \delta_k'$, then by the inductive hypothesis there is a non-empty portion $\delta^* \subset \delta_k'$ and an index $l \neq k$ such that for any continuous functions $f_1(t), \ldots, f_{k-1}(t), f_{k+1}(t), \ldots, f_m(t)$ there are continuous functions $f_1^*(t), \ldots, f_{k-1}^*(t), f_{k+1}^*(t), \ldots, f_m^*(t)$ such that

$$\sum_{i \neq k} f_i(q(x, y))\, p_i(x, y) = \sum_{i \neq k, l} f_i^*(q(x, y))\, p_i(x, y).$$

on the set $q^{-1}(\delta^*) \cap D$. Putting $f_k^*(t) = f_k(t)$, we obtain

$$\sum_{i=1}^{m} f_i(q(x, y))\, p_i(x, y) = \sum_{i \neq l} f_i^*(q(x, y))\, p_i(x, y).$$

So in case 1) the lemma is proved.

2) None of the sets δ_k contains non-empty portions of the set $q(D)$, that is, $\bigcup_{k=1}^{m} \delta_k$ is nowhere dense in $q(D)$. Therefore there exists a non-empty portion $\delta^* \subset \delta \setminus \bigcup_{k=1}^{m} \delta_k$. Since $\lambda(t, D, q, \{p_k\}) \equiv 0$ on δ^*, for every $t \in \delta^*$ there are numbers $\{C_k(t)\}$ $(\max_k |C_k(t)| = 1)$ such that $\sum_{k=1}^{m} C_k$

$(q(x, y)) p_k(x, y) \equiv 0$ on $e(q, t) \cap D$. If we had $C_k(t) = 0$ for some k, then it would turn out that $t \in \delta_k$. Consequently, $C_k(t) \neq 0$ for any k. We show that for every $t \in \delta^*$ the numbers $\{ C_k(t) \}$ are uniquely determined. Assume the contrary. Then there are numbers $\{ C_k'(t) \}$ $(\max | C_k'(t) | = 1)$ such that $\sum_{k=1}^{m} C_k'(q(x, y)) p_k(x, y) = 0$ on $e(q, t) \cap D$ and $C_k \neq C_k'$ for some k. Then

$$\sum_{k \neq 1} [C_k(t) C_1'(t) - C_k'(t) C_1(t)] p_k(x, y) = \sum_{k \neq 1} C_k''(t) p_k(x, y) \equiv 0$$

on $e(q, t) \cap D$ and in addition, $C_k'' \neq 0$ for some k. Consequently, $t \in \delta_1$. So we have obtained a contradiction, and the uniqueness of the choice of the numbers $C_k(t)$ is proved. Further, we may regard $\{ C_k(t) \}$ as single-valued functions of t on the portion δ^*. By Lemma 4.3.3, the functions $C_k(t)$ are continuous and, as noted above, $C_k(t) \neq 0$ for any $t \in \delta^*$. Then

$$p_1(x, y) = \sum_{k=2}^{m} - \frac{C_k(q(x, y))}{C_1(q(x, y))} p_k(x, y), \quad (x, y) \in q^{-1}(\delta^*) \cap D.$$

Putting $f(t) = f_k(t) - \dfrac{C_k(t)}{C_1(t)} f_1(t)$, $t \in \delta^*$, we have $\sum_{k=2}^{m} f_k^*(q(x, y)) p_k(x, y)$

$$= \sum_{k=1}^{m} f_k(q) p_k(x, y) - \sum_{k=2}^{m} \frac{C_k(q)}{C_1(q)} p_k(x, y)$$

$$= \sum_{k=2}^{m} f_k(q) p_k(x, y) + f_1(q) p_1(x, y)$$

$$= \sum_{k=1}^{m} f_k(q(x, y)) p_k(x, y), \quad (x, y) \in q^{-1}(\delta^*) \cap D.$$

This proves the lemma.

§ 4. *Reduction of linear superpositions to a form with independent terms*

We fix the continuous functions $p_i^k(x, y)$ and continuously differentiable functions $q_i(x, y)$ $(i = 0, 1, 2, \ldots, n;\ k = 1, 2, \ldots, m_i)$ $n \geqslant 2$, where $\{ q_i(x, y) \}$ satisfy in D conditions (1) and (3) of Lemma 4.2.2, and we consider in D superpositions of the form

$$\sum_{i=0}^{n} \sum_{k=1}^{m_i} p_i^k(x, y) f_i^k(q_i(x, y)),$$

where $\{ f_i^k(t) \}$ are arbitrary continuous functions of one variable.

We call a bounded closed region $G \subset D$ polyhedral if the boundary of G consists of a finite number of mutually non-intersecting simple closed contours that are unions of a finite number of segments of level curves of the functions $q_i(x, y)$ $(i=1, 2, \ldots, n)$. Let $G \subset D$ be a polyhedral region. We denote by Γ_i the set of those $t \in q_i(G)$ for which the set $e(q_i, t) \cap G$ contains a segment of a level curve belonging to the boundary of G. For any i the set Γ_i consists of a finite number of points. By property (1) of the functions $\{q_i(x, y)\}$ for every i and for all points $t_0 \in q_i(G) \backslash \Gamma_i$ there exists $\lim\limits_{t \to t_0} e(q_i, t) \rightleftharpoons e(q_0, t_0)$. If $t_0 \in \Gamma_i$, then the last assertion need not hold, but in any case there exists $\lim\limits_{t \to +t_0} e(q_i, t) \subset e(q_i, t_0)$ and $\lim\limits_{t \to -t_0} e(q_i, t) \subset e(q_i, t_0)$ where the limit is taken over the points $t \in q_i(G)$. Here the limit is understood in the sense of the distance $\rho\left(e(q_i, t), e(q_i, t_0)\right)$.

LEMMA 4.4.1. *There is a region* $G \subset D$ *and a system of numbers* $\tau_i^k = 0$ *or* 1 $(i=0, 1, 2, \ldots, n;\ k=1, 2, \ldots, m_i)$ *such that*

(4) *for any* i *and for any continuous functions* $\{\varphi_i^k(t)\}$ *there exist continuous functions* $\{f_i^k(t)\}$ *such that in* G

$$\sum_{k=1}^{m_i} p_i^k(x, y)\, \varphi_i^k(q_i(x, y)) \equiv \sum_{k=1}^{m_i} \tau_i^k p_i^k(x, y) f_i^k(q_i(x, y));$$

(5*) *for any polyhedral region* $G^* \subset G$ *and any* i, *the set*

$$\{t : \lambda(t, G^*, q_i, p_i^{k_1}, \ldots, p_i^{k_s}) = 0\}$$

is nowhere dense in $q_i(G^*)$, *where*

$$k_1 = k_1(i), k_2 = k_2(i), \ldots, k_s = k_s(i)$$

is the set of all values of k *for which* $\tau_i^k = 1$.

Proof. If $i = 0$, then by (1) the set $q_0(D)$ consists of only one point. We choose a region $G_0 \subset D$ and number τ_0^k $(k=1, 2, \ldots, m_0)$ such that in G_0 the functions $p_0^{k_1}, \ldots, p_0^{k_s}$ are a basis for the linear hull of the functions $\{p_0^k\}$ (condition (4) for $i=0$) and in any region $G^* \subset G_0$ these functions are linearly independent (condition (5*) for $i=0$). Let $G^* \subset D$ be an arbitrary polyhedral region. Then $\lambda(t, G^*, q, \{p_i^k\})$ as a function of t has, for any $i > 0$, a finite number of points of discontinuity (of the first kind) on the set $q_i(G^*)$, which consists of a finite number of segments (see Lemma 4.3.1). Hence it follows that if the set $\{t : \lambda(t, G^*, q_i, \{p_i^k\}) = 0\}$ is not nowhere dense on $q_i(G^*)$, then the function $\lambda(t) \equiv 0$ on some segment $\delta \subset q_i(G^*)$ not containing points of Γ_i. By Lemma 4.3.4, there is a segment $\delta^* \subset \delta$ such that in the expression $\sum_{k=1}^{m_i} p_i^k(x, y) f_i^k(q_i(x, y))$ one of the terms can be deleted, without narrowing the class of the functions representable in the region $q^{-1}(\delta^*) \cap G^*$ as superpositions of the given form. Carrying out all possible deletions we can find a region $G \subset G_0 \subset D$ for which the assertion of the lemma is satisfied.

A region $G \subset D$ is called regular if, firstly, it is polyhedral and, secondly, there is a number $\gamma_G > 0$ such that for every $i > 0$ and every $t \in q_i(G)$ the set $e(q_i, t) \cap G$ is the union of a finite number of simple arcs, each of which has length not less than γ_G. A point A of the boundary of the polyhedral region G is called a vertex if it belongs simultaneously to two segments of the level curves of $q_i(x, y)$ and $q_j(x, y)$ $(i \neq j)$ on the boundary of G. Every polyhedral region has a finite number of vertices.

LEMMA 4.4.2. *For every polyhedral region G and every neighbourhood U of the vertices of this region we can construct a regular region $G^* \subset G$ such that $G \backslash U \subset G^*$.*

Proof. Let $A_1, A_2, \ldots, A_r$ be the vertices of the polyhedral region G; $U_1, U_2, \ldots, U_r$ suitably small neighbourhoods of these vertices. Let $k_m = k_m(A_m)$ be the number of all those functions $\{q_i(x, y)\}$ for each of which the level curve passing through the point A_m does not contain any other points of the set $U_m \cap G$. Let $q_{im}(x, y)$ be one of these functions. We put $k(G) \in q_i(G)$. If $k(G) = 0$, then for any i and any $t \in q_i(G)$ the length of any component of the set $e(q_i, t) \cap G$ is greater than zero and consequently the region G is regular. Suppose that $k(G) > 0$ and m such that $k_m \neq 0$.

We fix $\varepsilon > 0$ and put

$$G_{1m}^* = G \,|\, \{(x, y): |q_{i_m}(x, y) - q(A_m)| < \varepsilon\} \cap U_m .$$

If U_m and ε are sufficiently small, then inside U_m the region G_{1m}^* has two vertices A_m' and A_m'', while the region G has only one vertex A_m there, but $k_m(A_m') = k_m(A_m'') = k_m(A_m) - 1$. We now put $G_1^* = \cap G_{1m}^*$, where the intersection is taken over all m such that $k_m \neq 0$. Then $k(G_1^*) = k(G) - 1$. Repeating this construction $k(G)$ times, we obtain a polyhedral region G^* for which $G \backslash G^* \subset U$ and $k(G^*) = 0$. Consequently, G^* is regular. This proves the lemma.

LEMMA 4.4.3. *There exists a set* $G \subset D$, *a number* $\lambda > 0$, *and a set of numbers* $\tau_i^k = 0$ *or* 1 $(i = 0, 1, \ldots, n;\ k = 1, 2, \ldots, m_i)$ *such that condition* (4) *of Lemma* 4.4.1 *is satisfied, and also the conditions*

(5) *for every* i *and* $t \in q_i(G)$ *and for any functions* $\{f_i^k(t)\}$

$$\max_{(x,y) \in e(q_i, t) \cap G} \left| \sum_{k=1}^{m_i} \tau_i^k p_i^k(x, y) f_i^k(q_i(x, y)) \right| \geqslant \lambda \max_k \left| \tau_i^k f_i^k(t) \right|;$$

(6) G *is a regular region.*

Proof. By Lemma 4.4.1 there exists a region $G^* \subset D$ and a set of numbers τ_i^k such that for every polyhedral subregion $G^{**} \subset G^*$ and for every i the set $\{t : \lambda(t, G^{**}, q_i, p_i^{k_1}, \ldots, p_i^{k_s}) = 0\}$ is nowhere dense in $q_i(G^{**})$, where $k_1, k_2, \ldots, k_s$ is the set of all values of k for which $\tau_i^k = 1$; moreover, on the set G^*, for any i the property (4) of Lemma 4.4.1 is satisfied. In order not to change the notation unnecessarily, we assume that all $\tau_i^k = 1$. We now construct a system of regular regions $G_0 \supset G_1 \supset G_2 \supset \ldots \supset G_n = G$, having the following property: for every $j \leqslant i$, $\inf_{t \in q_j(G_i)} \lambda(t, G_i, q_j, \{p_j^k\}) \geqslant \lambda_i > 0$. For G_0 we choose any regular region $G_0 \in G^*$. Suppose that the regular regions $G_0, G_1, \ldots, G_{i-1}$ have been constructed. We now construct the set G_i. We denote by α_δ the set $\{t : \lambda(t, q_i, G_{i-1}, \{p_i^k\}) > \delta\}$. Since the functions $\lambda(t, q_i, G_{i-1}, \{p_i^k\})$, have only finitely many points of discontinuity (of the first kind) on the set $q_i(G_{i-1})$, which consists of a finite number of segments (see Lemma 4.3.1), any component of α_δ is either an interval, or a half-interval, or a segment, or a point. Suppose that the set $\alpha_\delta^N \subset \alpha_\delta$ consists of the N longest components of non-zero length of the set α_δ (if α_δ has only N_0 $(< N)$ components of non-zero length, then let $\alpha_\delta^N = \alpha_\delta^{N_0}$). We denote by $\bar{\alpha}_\delta^N$ the closure of the set α_δ^N. We put $G_{i-1}^* = G_{i-1} \cap q_i^{-1}(\bar{\alpha}_\delta^N)$. We fix $\varepsilon > 0$. Since G_{i-1} is regular, for every j the length of any component of $e(q_j, t) \cap G_{i-1}$ is greater than $\gamma_G > 0$. And since the set $\{t : \lambda(t, q, G_{i-1}, \{p_i^k\}) = 0\}$ is nowhere dense in $q_i(G_{i-1})$, for sufficiently small δ and sufficiently large N the set G_{i-1}^* forms a $\varepsilon/2$-net on every set $e(q_j, t) \cap G_{i-1}$, $j < i$. The set G_{i-1}^* is a polyhedral region. We denote by $U(\varepsilon)$ the set of points (x, y) each of which is at a distance of no more than $\varepsilon/4$ from one of the vertices of the set G_{i-1}^*. By Lemma 4.4.2 there exists a regular region $G_i \subset G_{i-1}^*$ such that $G_{i-1}^* \backslash G_i \subset U(\varepsilon)$. The set G_i forms an ε-net on every set $e(q_j, t) \cap G_{i-1}, j < i$ and forms an $\varepsilon/2$-net on every set $e(q_i, t) \cap G_{i-1}^*$. By Lemma 4.3.2, for sufficiently small ε,

$$\lambda_i = \min_{j \leqq i} \inf_{t \in q_j(G_i)} \lambda(t, G_i, q_j, \{p_i^k\}) > \frac{1}{2} \min \left\{ \frac{\delta}{2}, \min_{j<i} \lambda_j \right\}.$$

Thus, the regular regions $G_1, G_2, \ldots, G_n$ can be constructed. The regular region $G = G_n$ satisfies all the requirements of our lemma ($\lambda = \lambda_n$), which is now proved.

§ 5. *The set of linear superpositions in the space of continuous functions is closed*

THEOREM 4.5.1. *Suppose that continuous functions $p_m(x, y)$ and continuously differentiable functions $q_m(x, y)$ ($m = 1, 2, \ldots, N$) are fixed. Then in any region D of the plane of the variables x, y. there exists a closed subregion $G \subset D$ such that the set of superpositions of the form*

$$\sum_{m=1}^{N} p_m(x, y) f_m(q_m(x, y)),$$

where $\{f_m(t)\}$ are arbitrary continuous functions, is closed (in the uniform metric) in the set of all functions continuous on the set G.

By Lemma 4.2.2 and 4.4.3 we can find a subset $G \subset D$, determine constants $\gamma > 0$ and $\lambda > 0$, and renumber the functions $\{p_m(x, y)\}$ and $\{q_m(x, y)\}$ with two indices so that the functions obtained after the renumbering, $\{p_i^k(x, y)\}$ and $\{q_i^k(x, y)\}$ ($i = 0, 1, 2, \ldots, n$; $k = 1, 2, \ldots, m_i$; $\sum_{i=0}^{n} m_i \leqslant N$) that is, some functions may be omitted in the renumbering) satisfy conditions (1), (2), (3) of Lemma 4.2.2, and also the conditions:

(4′) for any continuous functions $\{f_m(t)\}$ there exists continuous functions $\{f_i^k(t)\}$ such that on G

$$\sum_{m=1}^{N} p_m(x, y) f_m(q_m(x, y)) = \sum_{i=0}^{n} \sum_{k=1}^{m_i} p_i^k(x, y) f_i^k(q_i^k(x, y));$$

(5′) for every i and $t \in q_i^1(G)$ and for any functions $\{f_i^k(t)\}$

$$\max_{(x,y) \in e(q_i^1, t) \cap G} \left| \sum_{k=1}^{m_i} p_i^k(x, y) f_i^k(q_i^1(x, y)) \right| \leqslant \lambda \max_k |f_i^k(t)|;$$

(6′) G is a regular region with respect to the functions $\{q_i^k(x, y)\}$.

LEMMA 4.5.1. *In the sets* $\{ q_i^1 (G) \}$ *we can select subsets consisting of a finite number of points* $t_{i,j} \in q_i^1 (G)$ $(i = 0, 1, 2, ..., n;\ j = 1, 2, ..., s_i)$ *such that for any continuous functions* $\{ f_i^k (t) \}$

$$\max_{i,k} \max_{t \in q_i^1 (G)} |f_i^k (t)| \leq c \left(\max_{(x,y) \in G} \left| \sum_{i=0}^{n} \sum_{k=1}^{m_i} p_i^k (x, y) f_i^k (q_i^1 (x, y)) \right| + \max_{k} |f_i^k (t_{i,j})| \right),$$

where C *is a constant not depending on the functions* $\{ f_i^k (t) \}$.

Proof. Since G is polyhedral, for each i we can choose in $q_i (G)$ a finite set of points $\{ t_{i,j} \}$ so dense that the components of the level curves $e (q_i^1, t_{i,j}) \cap G$ form a δ-net in the set of all components of the level curves $e (q_i^1, t) \cap G$, $t \in q_i^1 (G)$. A sufficiently small δ, not depending on the functions $\{ f_i^k (t) \}$, will be chosen below. We put

$$\mu = \max_{i,k} \max_{(x,y) \in G} |f_i^k (q_i^1 (x, y))| ;$$

$$\varepsilon_1 = \max_{(x,y) \in G} \left| \sum_{i=0}^{n} \sum_{k=1}^{m_i} p_i^k (x, y) f_i^k (q_i^1 (x, y)) \right| ; \quad \varepsilon_2 = \max_{k,i,j} |f_i^k (t_{i,j})| .$$

For definiteness, let $f_1^1 (q_1^1 (a)) = \mu$ at the point $a \in G$. By (5′) there exists a point $a' \in G$ such that $\left| \sum_{k=1}^{m_i} p_1^k (a') f_1^k (q_1^1 (a')) \right| \geq \lambda \mu$. Let $[a', a^*]$ be a segment of the level curve of the function $q_1^1 (x, y)$ with end-points at a' and a^* such that $h_1 ([a', a^*]) \geq \gamma G/2$ (see the definition of a regular region in § 4). On the arc $[a', a^*]$ we fix a point a'' such that $\omega (\alpha) \leq \dfrac{\lambda}{2m_1}$, where $\alpha = h_1 ([a', a''])$. Then on the segment $[a', a'']$ the function $\varphi_1 (x, y) = \sum_{k=1}^{m_1} p_1^k (x, y) f_1^k (q_1^1 (x, y))$ keeps'a constant sign and satisfies the inequality $| \varphi_1 (x, y) | \geq \lambda \mu / 2$. In fact, $| \varphi_1 (a') | \geq \lambda \mu$ at the point a', and for any point $s \in [a', a'']$

$$| \varphi_1 (s) - \varphi_1 (a') | = \left| \sum_{k=1}^{m_1} (p_1^k (s) - p_1^k (a')) f_1^k (a') \right| \leq m_1 \mu \omega (\alpha) \leq \frac{\lambda \mu}{2} .$$

Consequently,

$$\left| \int_{s \in [a', a'']} \varphi_1 (s) \, ds \right| \geq \frac{1}{2} \lambda \mu \alpha .$$

By construction there is an index j and a segment $[b', b'']$ of the level curve $e (q_1', t_{1,j}) \cap G$ such that $\rho ([a', a''], [b', b'']) < \delta$. We have

$$\Big| \int_{s \in [b', b'']} \varphi_1(s)\, ds \Big| \leqslant c_1 \varepsilon_2 \beta ,$$

where $\beta = h_1([b', b''])$, $C_1 = m_1 \max\limits_{k} \max\limits_{(x,y) \in G} |p_1^k(x, y)|$. And since α and β are commensurable (δ will be chosen small in comparison with α),

$$\Big| \int_{s \in [a', a'']} \varphi_1(s)\, ds - \int_{s \in [b', b'']} \varphi_1(s)\, ds \Big| \geqslant \frac{1}{2} \lambda\mu\alpha - c_1' \varepsilon_2 \alpha .$$

By Lemma 4.2.3

$$\Big| \int_{s \in [a', a'']} \varphi_1(s)\, ds - \int_{s \in [b' b'']} \varphi_1(s)\, ds \Big| \leqslant c_3 (\alpha\varepsilon_1 + \mu\alpha\omega(\delta) + \mu\delta) .$$

Thus, $c_3(\alpha\varepsilon_1 + \mu\alpha\omega(\delta) + \mu\delta) \geqslant \lambda\mu\alpha/2 - c_1' \alpha \cdot \varepsilon_2$. If δ is taken sufficiently small in comparison with α (in order that $c_3(\alpha\omega(\delta) + \delta) < \lambda\alpha/2$), then we have $\mu \leqslant C(\varepsilon_1 + \varepsilon_2)$. This proves the lemma.

Let B be the Banach space consisting of all systems of functions $\{f_i^k(t)\}$, defined and continuous on the sets $\{q_i^1(G)\}$, with the norm

$$\| \{f_i^k(t)\} \|_B = \max_{i,k} \max_{t \in q_i^1(G)} |f_i^k(t)| \quad (i = 0, 1, 2, \ldots, n;\ k = 1, 2, \ldots, m_i) .$$

We denote by $C(G)$ the space of all functions $f(x, y)$ continuous on G with the uniform metric:

$$\| f(x, y) \|_{C(G)} = \max_{(x,y) \in G} |f(x, y)| .$$

LEMMA 4.5.2. *The linear operator* $T: B \to C(G)$ *acting by the formula*

$$T(\{f_i^k(t)\}) = f(x, y) = \sum_{i=0}^{n} \sum_{k=1}^{m_i} p_i^k(x, y) f_i^k(q_i^1(x, y)) ,$$

maps bounded closed sets of B *onto closed sets of* $C(G)$.

Proof. Let $F \subset B$ be a closed and bounded set of elements of B. Suppose that $f_n(x, y)$ is a sequence of functions in $T(F) \subset C(G)$, and that $f(x, y) \in C(G)$, where $\| f(x, y) - f_n(x, y) \|_{C(G)} \to 0$ as $n \to \infty$. We show that then $f(x, y) \in T(F)$. Since $f_n(x, y) \in T(F)$, there exists a sequence of elements $\{f_{i,n}^k(t)\} \in F$ such that $T(\{f_{i,n}^k(t)\}) = f_n(x, y)$. By Lemma 4.5.1 we can select in the sets $\{q_i^1(G)\}$ subsets consisting of a finite number of points $t_{i,j} \in q_i'(G)$ $(i = 0, 1, \ldots, n;\ j = 1, 2, \ldots, s_i)$ such that for each element $\{f_i^k(t)\} \in B$ the inequality

$$\| \{f_i^k(t)\} \|_B \leqslant c \big(\| f(x, y) \|_{C(G)} + \max_{k,j,i} |f_i^k(t_{i,j})| \big) ,$$

is satisfied, where the constant C does not depend on the functions $\{f_i^k(t)\}$. Since F is a bounded set, there exists a subsequence of suffixes $n_1, n_2, \ldots$ such that for any $i = 0, 1, \ldots, n$; $k = 1, 2, \ldots, m_i$; $j = 1, 2, \ldots, s_i$ the numerical sequence $f_{i,n_\nu}^k \to C_{k,i,j}$ as $\nu \to \infty$. From this and the previous inequality it follows that $\{f_{i,n_\nu}^k(t)\} \in F\,(\nu = 1, 2, \ldots)$ is a Cauchy sequence, because it is known that the sequence $f_n(x, y) \in T(F)$ is Cauchy sequence. Consequently there exists an element $\{f_i^k(t)\} \in B$ such that $\| \{f_i^k(t) - f_{i,n_\nu}^k(t)\} \|_B \to 0$. Since F is a closed set, $\{f_i^k(t)\} \in F$. The operator $T\colon B \to C(G)$ is bounded. Therefore $T(\{f_i^k(t)\}) = f(x, y)$. Consequently $f(x, y) \in T(F)$. This proves the lemma.

The following lemma from the theory of linear operators [28] turns out to be useful.

LEMMA 4.5.3. *Let B_1 and B_2 be Banach spaces. If a linear operator $T\colon B_1 \to B_2$ maps bounded closed sets of B_1 onto closed sets of B_2, then its domain of values is closed.*

Proof of Theorem 4.5.1. The set of superpositions of the form $\sum_{m=1}^{N} p_m(x, y) f_m(g_m(x, y))$ coincides on G with the set of superpositions of the form $\sum_{i=0}^{n} \sum_{k=1}^{m_i} p_i^k(x, y) f_i^k(q_i^1(x, y))$. By Lemma 4.5.2 and 4.5.3 the set of the latter superpositions is closed in the space $C(G)$. This proves the theorem.

§ 6. *The set of linear superpositions in the space of continuous functions is nowhere dense*

THEOREM 4.6.1. *For any continuous functions $p_m(x, y)$ and continuously differentiable functions $q_m(x, y)$ $(m = 1, 2, \ldots, N)$ and any region D of the plane of the variables x, y the set of superpositions of the form*

$$\sum_{m=1}^{N} p_m(x, y) f_m(q_m(x, y)),$$

where $\{f_m(t)\}$ are arbitrary continuous functions, is nowhere dense in the space of all functions continuous in D with uniform convergence.

By Lemma 4.2.2 we can find a subregion $G^* \subset D$, determine a constant $\gamma^* > 0$, and renumber the functions $\{q_m(x, y)\}$, with two indices so that the functions $\tilde{q}_i^k(x, y)$ $(i=0, 1, 2, ..., \tilde{n}; k=1, 2, ..., \tilde{m}_i; \sum_{i=0}^{\tilde{n}} \tilde{m}_i=N)$ obtained after the renumbering satisfy conditions (1), (2), (3) of Lemma 4.2.2. We now fix the point $(x_0, y_0) \in G^*$ and the number v so that the line $(y-y_0) + v(x-x_0) = 0$ does not touch at any of the level curves of the functions $\tilde{q}_i^k(x, y)$ $(i=1, 2, ..., \tilde{n})$ that pass through (x_0, y_0). Let $G^{**} \subset G^*$ be a disc with centre at (x_0, y_0) and radius small enough so that the $\{\tilde{q}_i^k(x, y)\}$ and $q_{N+1}(x, y) = y + vx$ satisfy condition (3) of Lemma 4.2.2 with some constant $\gamma^{**} > 0$. We put $p_{N+1}(x, y) = 1$. By Lemma 4.4.3 we can find a set $G \subset G^{**}$, determine a constant $\lambda > 0$, and again renumber the functions $p_m(x, y)$ and $q_m(x, y)$ $(m=1, 2, ..., N+1)$ with two indices so that the functions $p_i^k(x, y)$ and

$$q_i^k(x, y) \ (i=0, 1, 2, ..., n+1 \ ; \ k=1, 2, ..., m_i \ ; \sum_{i=0}^{n+1} m_i \leqslant N+1)$$

that is, some functions may be omitted in the renumbering) obtained after the renumbering satisfy conditions (1)-(3) of Lemma 4.2.2, conditions (4′)-(6′) of § 5, and the condition

$$7 \quad m_{n+1} = 1 \, , \ p_{N+1}^1 = p_{N+1}(x, y) = 1 \, , \ q_{N+1}^1 = q_{N+1}(x, y) = y + vx \, .$$

Let L be the linear space consisting of all system of functions $\{f_i^k(t)\}$ defined and continuous on the sets $\{q_i^1(G)\}$ and satisfying the condition

$$\sum_{i=0}^{n+1} \sum_{k=1}^{m_i} p_i^k(x, y) f_i^k(q_i^1(x, y)) \equiv 0 \quad \text{in} \quad G \, .$$

LEMMA 4.6.1. *L is a finite-dimensional linear space.*

Proof. By Lemma 4.5.1, in the sets $\{q_i^1(G)\}$ we can select a subset consisting of a finite number of points $\{t_{i,j}\}$ such that, if $\{f_i^k(t)\} \in L$ and $f_i^k(t_{i,j}) = 0$ for all k, i, j then $f_i^k(t) \equiv 0$ on $q_i^1(G)$ for all i, k. Thus, the set of functions $\{f_i^k(t)\}$ is completely determined by a finite set of parameters $\{f_i^k(t_{i,j})\}$. Consequently the dimension of the space L is finite. This proves the lemma.

LEMMA 4.6.2. *There exists a natural number μ such that in D the polynomial $(y+vx)^\mu = Q(x, y)$ is not equal to any superposition of the form $\sum_{m=1}^{N} p_m(x, y) f_m(q_m(x, y))$, where $\{f_m(t)\}$ are arbitrary continuous functions.*

Proof. We denote by Φ the space of functions of the form $f(y+vx) = f^{1}_{n+1}(q^{1}_{n+1}(x, y))$ that are representable on G by superpositions of the form $[\sum_{m=1}^{N} p_m(x, y) f_m(q_m(x, y))]$. Or, what comes to the same thing (see properties (4′) and (7)), of the form $[\sum_{i=0}^{n} \sum_{k=1}^{m_i} p_i^k(x, y) f^k_i(q^k_i(x, y))]$. Thus, functions of Φ satisfy the relation $\sum_{i=0}^{n+1} \sum_{k=1}^{m_i} p_i^k(x, y) f^k_i(q^k_i(x, y)) \equiv 0$ in G. Consequently the linear space Φ is naturally embedded in L. Since L is finite-dimensional (Lemma 4.6.1), Φ is also finite-dimensional. Let l be the dimension of Φ. Since the polynomials $(y+vx), (y+vx)^2, \ldots, (y+vx)^{l+1}$ are linearly independent, at least one of them $Q(x, y) = (y+vx)^{\mu}$ is not equal to any superposition of the form under discussion on G or, consequently, in D. This proves the lemma.

Proof of Theorem 4.6.1. By Lemma 4.6.2 the set of superpositions of the form given in Theorem 4.6.1 does not exhaust all continuous functions on G. Consequently, by Theorem 4.5.1, the set of these superpositions is a closed linear subspace of $C(G)$. Hence we conclude that the set of superpositions under discussion is nowhere dense in $C(G)$, nor consequently in $C(D)$. This proves the theorem.

COROLLARY 4.6.1. *For any continuous functions* $p_m(x_1, x_2, \ldots, x_n)$ *and continuously differentiable functions* $q_m(x_1, x_2, \ldots, x_n)$ $(m = 1, 2, \ldots, N)$ *and any region* D *of the space of the variables* $(x_1, x_2, \ldots, x_n)$ *the set of superpositions of the form*

$$\sum_{m=1}^{N} p_m(x_1, x_2, \ldots, x_n) f_m(q_m(x_1, x_2, \ldots, x_n), x_2, x_3, , x_{n-1}),$$

where $\{f_m(t, x_2, x_3, \ldots, x_{n-1})\}$ *are arbitrary continuous functions of* $(n-1)$ *variables, is nowhere dense in the space of all functions continuous in* D *with uniform convergence.*