

## **§2. Estimate of the difference of the integrals of one term of a superposition along nearby level curves**

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§ 2. *Estimate of the difference of the integrals of one term of a superposition along nearby level curves*

Let  $G$  be a region of the plane of the variables  $x$  and  $y$ , and  $q_1(x, y)$  and  $q_2(x, y)$  continuously differentiable functions satisfying in this region the following conditions: a) the partial derivatives with respect to  $x$  and with respect to  $y$  have modulus of continuity  $\omega(\delta)$ ; b) the inequalities

$$0 < \gamma \leq | \text{grad} [q_i(x, y)] | \leq \frac{1}{\gamma} < \infty \quad (i = 1, 2)$$

are satisfied everywhere in  $G$ , where  $\gamma$  is a constant; c) for any point  $(x, y) \in G$  the absolute value of the acute angle formed by the level curves of the functions  $q_1(x, y)$  and  $q_2(x, y)$  which pass through this point is greater than some positive constant  $\gamma$ .

LEMMA 4.2.1. *Let  $e'_{q_2}$  and  $e''_{q_2}$  be two level curves of the function  $q_2$  and  $e'_{q_1}$  and  $e''_{q_1}$  level curves of the function  $q_1$ ;  $[a', a''] \subset G$  the segment of the curve  $e'_{q_1}$  with end-points  $a' \in e'_{q_2}$  and  $a'' \in e''_{q_2}$ ;  $[b', b'']$  the segment of the curve  $e''_{q_1}$  with end-points  $b' \in e'_{q_2}$  and  $b'' \in e''_{q_2}$ . Then*

$$h_1([b', b'']) \leq h_1([a', a'']) \times (1 + c_1(\gamma) \omega(\delta)),$$

where  $\delta = d_1([a', a''] \cup [b', b''])$  and  $c_1(\gamma)$  depends only on  $\gamma$ .

*Proof.* Since  $q_2(a'') - q_2(a') = q_2(b'') - q_2(b')$ , we have

$$\int_{s \in [a', a'']} \frac{\partial q_2}{\partial s} ds = \int_{s \in [b', b'']} \frac{\partial q_2}{\partial s} ds.$$

Consequently,  $\frac{\partial q_2(a^*)}{\partial s} h_1([a', a'']) = \frac{\partial q_2(b^*)}{\partial s} h_1([b', b''])$ , where  $\frac{\partial q_2(a^*)}{\partial s}$

and  $\frac{\partial q_2(b^*)}{\partial s}$  are the derivatives at the points  $a^* \in [a', a'']$  and  $b^* \in [b', b'']$

along the curves  $[a', a'']$  and  $[b', b'']$ , respectively. We show that  $\frac{\partial q_2(a^*)}{\partial s}$

$= \frac{\partial q_2(b^*)}{\partial s} + O(\gamma) \omega(\delta)$ . We denote by  $q_2^*$  the derivative of  $q_2$  at the point  $b^*$

in the direction of  $\tau(e'_{q_1}, a^*)$  and put  $\alpha = \gamma \{ \tau[e''_{q_1}, b^*], \tau[e'_{q_1}, a^*] \}$ . From

conditions a) and b) it follows that  $\frac{\partial q_2(a^*)}{\partial s} = q_2^* + O(1) \omega(\delta)$  and  $\alpha$

$= O(\gamma) \omega(\delta)$ . We denote by  $\beta_1$  and  $\beta_2$  the values of the angles formed by the vectors  $\tau [e''_{q_1}, b^*]$  and  $\tau [e'_{q_1}, a^*]$  with the vector  $\text{grad} [q_2(b^*)]$ . We have

$$\left| q_2^* - \frac{\partial q_2(b^*)}{\partial s} \right| = |\text{grad} [q_2(b^*)]| |\cos \beta_2 - \cos \beta_1| = O(\gamma) \alpha = O(\gamma) \omega(\delta).$$

Thus,

$$\frac{\partial q_2(a^*)}{\partial s} = q_2^* + O(1) \omega(\delta) = \frac{\partial q_2(b^*)}{\partial s} + O(1) \left\{ \left| q_2^* - \frac{\partial q_2(b^*)}{\partial s} \right| + \omega(\delta) \right\} = \frac{\partial q_2(b^*)}{\partial s} + O(\gamma) \omega(\delta).$$

Consequently,

$$\begin{aligned} h_1([b', b'']) &= h_1([a', a'']) \frac{\partial q_2(a^*)}{\partial s} \left( \frac{\partial q_2(b^*)}{\partial s} \right)^{-1} \\ &= h_1([a', a'']) \left( 1 + O(\gamma) \omega(\delta) \left( \frac{\partial q_2(b^*)}{\partial s} \right)^{-1} \right) \\ &= h_1([a', a'']) (1 + O(\gamma) \omega(\gamma)), \end{aligned}$$

since by virtue of b)  $\frac{\partial q_2(b^*)}{\partial s} > |\text{grad} [q_2(b^*)]| \sin \gamma$ . This, proves the lemma.

LEMMA 4.2.2. Let  $q_m(x, y)$  ( $m=1, 2, \dots, N$ ) be continuously differentiable functions. In any region  $D$  we can find a subregion  $G \subset D$ , determine a constant  $\gamma > 0$ , and renumber the functions  $\{q_m(x, y)\}$  with two indices so that the functions

$$q_i^k(x, y) = q_m(x, y) \quad (i=0, 1, 2, \dots, n; k=1, 2, \dots, m_i; \sum_{i=0}^n m_i = N)$$

obtained after the renumbering satisfy the following conditions:

(1) when  $i=0$ ,  $q_i^k = \text{const}$  in  $G$ , and when  $i > 0$ ,  $\gamma \leq |\text{grad} [q_i^k(x, y)]| \leq \frac{1}{\gamma}$  for every point  $(x, y) \in G$ ;

(2) the functions  $q_i^k(x, y)$  ( $i > 0$  fixed,  $k=1, 2, \dots, m_i$ ) have in the region  $G$  identical sets of level curves, more precisely, in the region  $G$ ,  $q_i^k(x, y) \equiv \varphi_i^{k,l}(q_i^l(x, y))$ , where  $\varphi_i^{k,l}(t)$  is a strictly monotonic continuously differentiable function of  $t$ ;

(3) when  $i \neq j$  ( $i, j \neq 0$ ), then for any  $k$  and  $l$  the absolute value of the acute angle formed by the level curves of the functions  $q_i^k(x, y)$  and  $q_j^l(x, y)$  which pass through an arbitrary point  $(x, y) \in G$  is greater than  $\gamma$ .

*Proof.* By the continuity of the partial derivatives of the functions  $\{q_m(x, y)\}$  there exists a subregion  $G^* \subset D$  inside which for any function  $q_m(x, y)$  either  $\text{grad } q_m(x, y) \equiv 0$  or  $|\text{grad } q_m(x, y)|$  is greater than some positive constant. From the continuity of the partial derivatives of the functions  $\{q_m(x, y)\}$  it follows also that there exists a subregion  $G^{**} \subset G^*$  inside which for any pair of functions  $q_r(x, y)$  and  $q_s(x, y)$  one of two conditions holds: either  $D\left(\frac{q_r, q_s}{x, y}\right) \equiv 0$  in  $G^{**}$ , or for every point of  $G^{**}$  the level curves of  $q_r(x, y)$  and  $q_s(x, y)$  that pass through this point intersect at a non-zero angle ( $D\left(\frac{q_r, q_s}{x, y}\right) \neq 0$  in  $G^{**}$ ). From the implicit function theorem it follows that there exists a subregion  $G \subset G^{**}$  in which condition (2) is satisfied for every pair of functions  $q_r(x, y)$  and  $q_s(x, y)$  with gradients different from zero and with determinant  $D\left(\frac{q_r, q_s}{x, y}\right) \equiv 0$ .

We now renumber the functions  $\{q_m(x, y)\}$  with two indices in such a way that only functions constant in  $G$  have lower index zero, and the same lower index is assigned to those functions whose level curves coincide identically in  $G$ . This proves the lemma.

We consider in the region  $G$  a superposition of the form  $\sum_{i=0}^n \sum_{k=1}^{m_i} p_i^k(x, y) f_i^k(q_i^k(x, y))$ , where  $\{f_i^k(t)\}$  are continuous functions of one variable,  $\{p_i^k(x, y)\}$  are continuous functions satisfying in  $G$  the condition  $|p_i^k(x, y)| \leq \frac{1}{\gamma}$  and  $\{q_i^k(x, y)\}$  are continuously differentiable functions satisfying in  $G$  conditions (1), (2), (3) of Lemma 4.2.2. Let  $\omega(\delta)$  be the common modulus of continuity in  $G$  of the functions  $\left\{p_i^k(x, y); \frac{\partial q_i^k(x, y)}{\partial x}; \frac{\partial q_i^k(x, y)}{\partial y}\right\}$ . Let  $[a', a'']$  and  $[b', b'']$  be segments of the level curves of the functions  $\{q_i^k(x, y)\}$  ( $i > 0$  fixed) lying in  $G$ . Let

$$\alpha = h_1([a', a'']); \quad \delta = \rho([a', a''], [b', b'']);$$

$$\varepsilon = \sup \left| \sum_{i=0}^n \sum_{k=1}^{m_i} p_i^k(x, y) f_i^k(q_i^k(x, y)) \right|;$$

$$m = \max_{i,k} \sup |f_i^k(q_i^k(x, y))|,$$

where sup is taken over all points  $(x, y) \in [a', a''] \cup [b', b'']$ .

LEMMA 4.2.3. *If  $\delta$  is sufficiently small ( $\omega(\delta) \leq C_2(\gamma)$ ), then for any  $i > 0$*

$$\left| \int_{s \in [a', a'']} \sum_{k=1}^{m_i} p_i^k(s) f_i^k(q_i^k(s)) ds - \int_{s \in [b', b'']} \sum_{k=1}^{m_i} p_i^k(s) f_i^k(q_i^k(s)) ds \right| \leq C_3(\gamma)(\alpha\varepsilon + m\alpha\omega(\delta) + m\delta),$$

where the constants  $C_2(\gamma), C_3(\gamma)$  depend only on  $\gamma$ .

*Proof.* By (1), (2), (3) there exists a sufficiently small constant  $C_2(\gamma)$  and a sufficiently large constant  $C_3(\gamma)$  such that if  $\omega(\delta) \leq C_2(\gamma)$  and for a point  $a \in [a', a'']$  the inequalities  $h_1([a', a]) \geq C_3(\gamma)\delta; h_1([a, a'']) \geq C_3(\gamma)\delta$  are satisfied, then for any  $j \neq i$  ( $j > 0$ ) the level curve of the function  $q_j^k$  that passes through  $a$  intersects  $[b', b'']$  of the level curve of  $q_i^k$ . Suppose that  $\alpha > 2C_3(\gamma)\delta$  (if  $\alpha \leq 2C_3(\gamma)\delta$ , then the assertion of the lemma is trivial)

and suppose that the segment  $[\tilde{a}', \tilde{a}']$  of the level curve of  $q_i^k$  is such that  $[\tilde{a}', \tilde{a}'] \subset [a', a'']$  and  $h_1([\tilde{a}', \tilde{a}']) = h_1([\tilde{a}'', a'']) = C_3(\gamma)\delta$ . On the arc  $[\tilde{a}', \tilde{a}']$  we fix a system of points  $a_1, a_2, \dots, a_\nu$  ( $\tilde{a}' = a_1, \tilde{a}'' = a_\nu$ ), uniformly distributed along the length of this arc, and denote by  $b_r$  the point of intersection of  $[b', b'']$  with the level curve of  $q_j^k$  that passes through  $a_r$  (here  $j \neq i$  should for the time being be regarded as fixed). Using Lemma 4.2.1 we have

$$\begin{aligned} & \left| \int_{s \in [a', a'']} p_j^k(s) f_j^k(q_j^k(s)) ds - \int_{s \in [b', b'']} p_j^k(s) f_j^k(q_j^k(s)) ds \right| \\ &= \left| \int_{s \in [a_1, a_\nu]} p_j^k(s) f_j^k(q_j^k(s)) ds - \int_{s \in [b_1, b_\nu]} p_j^k(s) f_j^k(q_j^k(s)) ds \right| \\ &+ O(\gamma) m\delta \\ &= \lim_{\nu \rightarrow \infty} \left| \sum_{r=1}^{\nu} p_j^k(a_r) f_j^k(q_j^k(a_r)) h_1([a_r, a_{r+1}]) \right. \\ &\quad \left. - \sum_{r=1}^{\nu} p_j^k(b_r) f_j^k(q_j^k(b_r)) h_1([b_r, b_{r+1}]) \right| + O(\gamma) m\delta \end{aligned}$$

$$\begin{aligned}
 &= \lim_{v \rightarrow \infty} \left| \sum_{r=1}^v p_j^k(a_r) f_j^k(q_j^k(a_r)) h_1([a_r, a_{r+1}]) \right. \\
 &\quad - \sum_{r=1}^v p_j^k(a_r) f_j^k(q_j^k(a_r)) h_1([a_r, a_{r+1}]) (1 + O(\gamma) \omega(\delta)) \\
 &\quad \left. + \sum_{r=1}^v (p_j^k(a_r) - p_j^k(b_r)) f_j^k(q_j^k(a_r)) h_1([b_r, b_{r+1}]) \right| + O(\gamma) m \delta \\
 &= \lim_{v \rightarrow \infty} \left| \sum_{r=1}^v p_j^k(a_r) f_j^k(q_j^k(a_r)) h_1([a_r, a_{r+1}]) O(\gamma) \omega(\delta) \right. \\
 &\quad \left. + \sum_{r=1}^v f_j^k(q_j^k(a_r)) h_1([b_r, b_{r+1}]) O(\gamma) \omega(\delta) \right| + O(\gamma) m \delta \\
 &= O(\gamma) m \alpha \omega(\delta) + O(\gamma) m \alpha \omega(\delta) + O(\gamma) m \delta = O(\gamma) m (\delta + \alpha \omega(\delta)).
 \end{aligned}$$

Then

$$\begin{aligned}
 &\left| \int_{s \in [a', a'']} \sum_{k=1}^{m_i} p_i^k(s) f_i^k(q_i^k(s)) ds - \int_{s \in [b', b'']} \sum_{k=1}^{m_i} p_i^k(s) f_i^k(q_i^k(s)) ds \right. \\
 &\leq \left| \int_{s \in [a', a'']} \sum_{i=0}^n \sum_{k=1}^{m_i} p_i^k(s) f_i^k(q_i^k(s)) ds - \int_{s \in [b', b'']} \sum_{i=0}^n \sum_{k=1}^{m_i} p_i^k(s) f_i^k(q_i^k(s)) ds \right| \\
 &\quad + \left| \sum_{j \neq i} \int_{s \in [a', a'']} \sum_{k=1}^{m_j} p_j^k(s) f_j^k(q_j^k(s)) ds - \int_{s \in [b', b'']} \sum_{k=1}^{m_j} p_j^k(s) f_j^k(q_j^k(s)) ds \right| \\
 &\leq C_4(\gamma) \alpha \varepsilon + n (\max_{j \neq i} m_j) C_5(\gamma) m (\delta + \alpha \omega(\delta)) \\
 &\leq C_3(\gamma) (\alpha \varepsilon + m \delta + m \alpha \omega(\delta)).
 \end{aligned}$$

This proves the lemma.

### § 3. Deletion of dependent terms

On a bounded closed set  $D$  we consider the space of linear superpositions of the form  $\sum_{k=1}^m p_k(x, y) f_k(q(x, y))$ ,  $(x, y) \in D$ . Here the functions  $\{p_k(x, y)\}$  and  $q(x, y)$  are continuous and fixed, and  $\{f_k(t)\}$  are arbitrary continuous functions of one variable. We assume that the function  $q(x, y)$  is such that for any sequence  $t_n \in q(D) \rightarrow t \in q(D)$  we have  $\rho[e(q, t_n) \cap D, e(q, t) \cap D] \rightarrow 0$ . We put

$$\lambda(t, D, q, p_1, \dots, p_m) = \inf_{\{c_k\}} \sup_{(x, y) \in e(q, t) \cap D} \left| \sum_{k=1}^m c_k p_k(x, y) \right|,$$