

§3. Deletion of dependent terms

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$$\begin{aligned}
 &= \lim_{v \rightarrow \infty} \left| \sum_{r=1}^v p_j^k(a_r) f_j^k(q_j^k(a_r)) h_1([a_r, a_{r+1}]) \right. \\
 &- \sum_{r=1}^v p_j^k(a_r) f_j^k(q_j^k(a_r)) h_1([a_r, a_{r+1}]) (1 + O(\gamma) \omega(\delta)) \\
 &+ \left. \sum_{r=1}^v (p_j^k(a_r) - p_j^k(b_r)) f_j^k(q_j^k(a_r)) h_1([b_r, b_{r+1}]) \right| + O(\gamma) m \delta \\
 &= \lim_{v \rightarrow \infty} \left| \sum_{r=1}^v p_j^k(a_r) f_j^k(q_j^k(a_r)) h_1([a_r, a_{r+1}]) O(\gamma) \omega(\delta) \right. \\
 &+ \left. \sum_{r=1}^v f_j^k(q_j^k(a_r)) h_1([b_r, b_{r+1}]) O(\gamma) \omega(\delta) \right| + O(\gamma) m \delta \\
 &= O(\gamma) m \alpha \omega(\delta) + O(\gamma) m \alpha \omega(\delta) + O(\gamma) m \delta = O(\gamma) m (\delta + \alpha \omega(\delta)).
 \end{aligned}$$

Then

$$\begin{aligned}
 &\left| \int_{s \in [a', a'']} \sum_{k=1}^{m_i} p_i^k(s) f_i^k(q_i^k(s)) ds - \int_{s \in [b', b'']} \sum_{k=1}^{m_i} p_i^k(s) f_i^k(q_i^k(s)) ds \right. \\
 &\leq \left| \int_{s \in [a', a'']} \sum_{i=0}^n \sum_{k=1}^{m_i} p_i^k(s) f_i^k(q_i^k(s)) ds - \int_{s \in [b', b'']} \sum_{i=0}^n \sum_{k=1}^{m_i} p_i^k(s) f_i^k(q_i^k(s)) ds \right| \\
 &+ \left| \sum_{j \neq i} \int_{s \in [a', a'']} \sum_{k=1}^{m_j} p_j^k(s) f_j^k(q_j^k(s)) ds - \int_{s \in [b', b'']} \sum_{k=1}^{m_j} p_j^k(s) f_j^k(q_j^k(s)) ds \right| \\
 &\leq C_4(\gamma) \alpha \varepsilon + n (\max_{j \neq i} m_j) C_5(\gamma) m (\delta + \alpha \omega(\delta)) \\
 &\leq C_3(\gamma) (\alpha \varepsilon + m \delta + m \alpha \omega(\delta)).
 \end{aligned}$$

This proves the lemma.

§ 3. Deletion of dependent terms

On a bounded closed set D we consider the space of linear superpositions of the form $\sum_{k=1}^m p_k(x, y) f_k(q(x, y))$, $(x, y) \in D$. Here the functions $\{p_k(x, y)\}$ and $q(x, y)$ are continuous and fixed, and $\{f_k(t)\}$ are arbitrary continuous functions of one variable. We assume that the function $q(x, y)$ is such that for any sequence $t_n \in q(D) \rightarrow t \in q(D)$ we have $\rho[e(q, t_n) \cap D, e(q, t) \cap D] \rightarrow 0$. We put

$$\lambda(t, D, q, p_1, \dots, p_m) = \inf_{\{c_k\}} \sup_{(x, y) \in e(q, t) \cap D} \left| \sum_{k=1}^m c_k p_k(x, y) \right|,$$

where inf is taken over all sets of numbers $\{c_k\}$ for which $\max_k |c_k| = 1$. The function $\lambda(t, D, q, \{p_k\})$, as a function of t , is defined only on the set $q(D)$.

LEMMA 4.3.1. *The function $\lambda(t, D, q, \{p_k\})$ depends continuously on t .*

Proof. The linear combinations $\sum_{k=1}^m c_k p_k(x, y)$ for all possible systems of numbers $\{c_k\}$ for which $\max_k |c_k| \leq 1$, form an equicontinuous set of functions, considered on the bounded closed set D . Consequently, for any $\varepsilon > 0$ there is a $\delta > 0$ such that if $|t_1 - t_2| < \delta$, then

$$\left| \sup_{(x, y) \in e(q, t_1)} \left| \sum_{k=1}^m c_k p_k(x, y) \right| - \sup_{(x, y) \in e(q, t_2)} \left| \sum_{k=1}^m c_k p_k(x, y) \right| \right| < \varepsilon$$

simultaneously for all systems of numbers $\{c_k\}$ such that $\max_k |c_k| \leq 1$.

For definiteness, suppose that $\lambda(t_2, D, q, \{p_k\}) \geq \lambda(t_1, D, q, \{p_k\})$.

Since the expression $\sup_{(x, y) \in e(q, t_1)} \left| \sum_{k=1}^m c_k p_k(x, y) \right|$ depends continuously on the coefficients $\{c_k\}$, there exists a system of numbers $\{c_k^1\}$ such that $\max_k |c_k^1| = 1$ and

$$\lambda(t_1, D, q, \{p_k\}) = \sup_{(x, y) \in e(q, t_1)} \left| \sum_{k=1}^m c_k^1 p_k(x, y) \right|.$$

Since

$$\lambda(t_2, D, q, \{p_k\}) \leq \sup_{(x, y) \in e(q, t_2)} \left| \sum_{k=1}^m c_k^1 p_k(x, y) \right|,$$

we have

$$0 \leq \lambda(t_2) - \lambda(t_1) \leq \sup_{(x, y) \in e(q, t_2)} \left| \sum_{k=1}^m c_k^1 p_k(x, y) \right|$$

$$- \sup_{(x, y) \in e(q, t_1)} \left| \sum_{k=1}^m c_k^1 p_k(x, y) \right| < \varepsilon.$$

This proves the lemma.

LEMMA 4.3.2. *The function $\lambda(t, D, q, \{p_k\})$ depends continuously on D in the sense that there exists a function $\mu(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$, having the property: if the set $D_\varepsilon \subset D$ is such that, for any t , $D_\varepsilon \cap e(q, t)$ forms an ε -net in the set $e(q, t) \cap D$, then*

$$\max_{t \in q(D)} \left| \lambda(t, D, q, \{p_k\}) - \lambda(t, D_\varepsilon, q, \{p_k\}) \right| \leq \mu(\varepsilon).$$

Proof. Using the equicontinuity of the set of functions $\sum_{k=1}^n c_k p_k(x, y)$ where $\max_k |c_k| \leq 1$, we conclude that there exists a function $\mu(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$ such that the inequality

$$0 \leq \sup_{(x, y) \in e(q, t) \cap D} \left| \sum_{k=1}^m c_k p_k(x, y) \right| - \sup_{(x, y) \in e(q, t) \cap D_\varepsilon} \left| \sum_{k=1}^m c_k p_k(x, y) \right| \leq \mu(\varepsilon).$$

uniformly over all $t \in q(D)$ and over all systems of numbers $\{c_k\}$ for which $\max_k |c_k| \leq 1$. For any $\varepsilon > 0$ there exists a system of numbers $\{c_k^\varepsilon\}$ such that $\max_k |c_k^\varepsilon| = 1$ and

$$\lambda(t, D_\varepsilon, q, \{p_k\}) = \sup_{(x, y) \in e(q, t) \cap D_\varepsilon} \left| \sum_{k=1}^m c_k^\varepsilon p_k(x, y) \right|.$$

Since for any ε

$$\lambda(t, D, q, \{p_k\}) \leq \sup_{(x, y) \in e(q, t) \cap D} \left| \sum_{k=1}^m c_k^\varepsilon p_k(x, y) \right|$$

and, on the other hand, $\lambda(t, D, q, \{p_k\}) \geq \lambda(t, D_\varepsilon, q, \{p_k\})$ (we recall that $D_\varepsilon \subset D$), we have

$$0 \leq \lambda(t, D, q, \{p_k\}) - \lambda(t, D_\varepsilon, q, \{p_k\}) \leq \sup_{(x, y) \in e(q, t) \cap D} \left| \sum_{k=1}^m c_k^\varepsilon p_k(x, y) \right| - \sup_{(x, y) \in e(q, t) \cap D_\varepsilon} \left| \sum_{k=1}^m c_k^\varepsilon p_k(x, y) \right| < \mu(\varepsilon).$$

This proves the lemma.

LEMMA 4.3.3. Let F be a closed set on the t -axis; $F \subset q(D)$. For every $t \in F$, suppose that there exists one and only one system of numbers $\{C_k\}$ ($\max_k |C_k| = 1$) such that $\sum_{k=1}^m C_k p_k(x, y) \equiv 0$ on the set $e(q, t) \cap D$. Then each of the functions $\{C_k(t)\}$ depends continuously on t on the set F .

Proof. Suppose that $t_n \in F, t \in F$ and $t_n \rightarrow t$. We put $\overline{\lim}_{n \rightarrow \infty} C_k(t_n) = \tilde{C}_k$ and $\underline{\lim}_{n \rightarrow \infty} C_k(t_n) = \tilde{\tilde{C}}_k$. Since $\sum_{k=1}^m C_k(t_n) p_k(x, y) \equiv 0$ on the set $e(q, t_n) \cap D$ and $\rho[e(q, t) \cap D, e(q, t_n) \cap D] \rightarrow 0$ as $n \rightarrow \infty$, we have $\sum_{k=1}^m \tilde{C}_k p_k(x, y)$

$\equiv 0 \equiv \sum_{k=1}^m \tilde{C}_k p_k(x, y)$ on the set $e(q, t) \cap D$. Consequently, by the condition of the lemma, $\tilde{C}_k = \tilde{C}_k = C_k(t)$. This proves the lemma.

LEMMA 4.3.4. *Suppose that $\lambda(t, D, q, \{p_k\}) \equiv 0$ on some non-empty portion δ of the set $q(D)$. Then there is a non-empty portion $\delta^* \subset \delta$ and an index l such that for any continuous functions $\{f_k(t)\}$ there are continuous functions $\{f_k^*(t)\}$ such that*

$$\sum_{k \neq l} f_k^*(q(x, y)) p_k(x, y) = \sum_{k=1}^m f_k(q(x, y)) p_k(x, y)$$

on the set $q^{-1}(\delta^*) \cap D$.

We recall that a portion δ of a set E is that part of it which lies in the interval δ .

Proof. We prove the lemma by induction on m . For $m = 1$ the assertion of the lemma is obvious. We denote by δ_k the set of all points t of the portion δ for which $\lambda(t, D, q, p_1, \dots, p_{k-1}, p_{k+1}, \dots, p_m) = 0$. By Lemma 4.3.1, the set is closed. Two cases are possible.

1) For some k the set δ_k contains a non-empty portion δ'_k of the set $q(D)$. Since $\lambda(t, D, q, p_1, \dots, p_{k-1}, p_{k+1}, \dots, p_m) = 0$ for every $t \in \delta'_k$, then by the inductive hypothesis there is a non-empty portion $\delta^* \subset \delta'_k$ and an index $l \neq k$ such that for any continuous functions $f_1(t), \dots, f_{k-1}(t), f_{k+1}(t), \dots, f_m(t)$ there are continuous functions $f_1^*(t), \dots, f_{k-1}^*(t), f_{k+1}^*(t), \dots, f_m^*(t)$ such that

$$\sum_{i \neq k} f_i(q(x, y)) p_i(x, y) = \sum_{i \neq k, l} f_i^*(q(x, y)) p_i(x, y).$$

on the set $q^{-1}(\delta^*) \cap D$. Putting $f_k^*(t) = f_k(t)$, we obtain

$$\sum_{i=1}^m f_i(q(x, y)) p_i(x, y) = \sum_{i \neq l} f_i^*(q(x, y)) p_i(x, y).$$

So in case 1) the lemma is proved.

2) None of the sets δ_k contains non-empty portions of the set $q(D)$, that is, $\bigcup_{k=1}^m \delta_k$ is nowhere dense in $q(D)$. Therefore there exists a non-empty portion $\delta^* \subset \delta \setminus \bigcup_{k=1}^m \delta_k$. Since $\lambda(t, D, q, \{p_k\}) \equiv 0$ on δ^* , for every $t \in \delta^*$ there are numbers $\{C_k(t)\}$ ($\max_k |C_k(t)| = 1$) such that $\sum_{k=1}^m C_k$

$(q(x, y)) p_k(x, y) \equiv 0$ on $e(q, t) \cap D$. If we had $C_k(t) = 0$ for some k , then it would turn out that $t \in \delta_k$. Consequently, $C_k(t) \neq 0$ for any k . We show that for every $t \in \delta^*$ the numbers $\{C_k(t)\}$ are uniquely determined. Assume the contrary. Then there are numbers $\{C'_k(t)\}$ ($\max |C'_k(t)| = 1$) such that $\sum_{k=1}^m C'_k(q(x, y)) p_k(x, y) = 0$ on $e(q, t) \cap D$ and $C_k \neq C'_k$ for some k . Then

$$\sum_{k \neq 1} [C_k(t) C'_1(t) - C'_k(t) C_1(t)] p_k(x, y) = \sum_{k \neq 1} C'_k(t) p_k(x, y) \equiv 0$$

on $e(q, t) \cap D$ and in addition, $C'_k \neq 0$ for some k . Consequently, $t \in \delta_1$. So we have obtained a contradiction, and the uniqueness of the choice of the numbers $C_k(t)$ is proved. Further, we may regard $\{C_k(t)\}$ as single-valued functions of t on the portion δ^* . By Lemma 4.3.3, the functions $C_k(t)$ are continuous and, as noted above, $C_k(t) \neq 0$ for any $t \in \delta^*$. Then

$$p_1(x, y) = \sum_{k=2}^m -\frac{C_k(q(x, y))}{C_1(q(x, y))} p_k(x, y), \quad (x, y) \in q^{-1}(\delta^*) \cap D.$$

Putting $f(t) = f_k(t) - \frac{C_k(t)}{C_1(t)} f_1(t)$, $t \in \delta^*$, we have $\sum_{k=2}^m f_k^*(q(x, y)) p_k(x, y)$

$$\begin{aligned} &= \sum_{k=1}^m f_k(q) p_k(x, y) - \sum_{k=2}^m \frac{C_k(q)}{C_1(q)} p_k(x, y) \\ &= \sum_{k=2}^m f_k(q) p_k(x, y) + f_1(q) p_1(x, y) \\ &= \sum_{k=1}^m f_k(q(x, y)) p_k(x, y), \quad (x, y) \in q^{-1}(\delta^*) \cap D. \end{aligned}$$

This proves the lemma.

§ 4. *Reduction of linear superpositions to a form with independent terms*

We fix the continuous functions $p_i^k(x, y)$ and continuously differentiable functions $q_i(x, y)$ ($i=0, 1, 2, \dots, n; k=1, 2, \dots, m_i$) $n \geq 2$, where $\{q_i(x, y)\}$ satisfy in D conditions (1) and (3) of Lemma 4.2.2, and we consider in D superpositions of the form

$$\sum_{i=0}^n \sum_{k=1}^{m_i} p_i^k(x, y) f_i^k(q_i(x, y)),$$

where $\{f_i^k(t)\}$ are arbitrary continuous functions of one variable.