§2. (,)-entropy of the set of linear superpositions

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Corollary 5.1.3. *For any continuous functions* p_i *and continuously differentiable functions* $q_{1,i}, q_{2,i}, \ldots, q_{k,i}, k < n$ $(i = 1, 2, \ldots, N)$ *and every region* G_n *there exists a continuous function that is not equal in* G_n *to any superposition of the form* (V).

§ 2. (ε, δ)-*entropy of the set of linear superpositions*

We denote by $S(\delta, z)$ the disc of radius δ with centre at z. Let $p(z) = p(x, y)$ and $q(z) = q(x, y)$ be functions defined in a closed region G of the x, y-plane and having the properties:

a) $p(x, y), \dfrac{\partial q(x, y)}{\partial x}, \dfrac{\partial q(x, y)}{\partial y}$ are continuous in G and have modulus of continuity $\omega(\delta)$,

b) the inequalities $0 < \gamma \leqslant |\operatorname{grad}[q(r)]| \leqslant \dfrac{1}{\gamma}$ and $|p(z)| \leqslant \dfrac{1}{\gamma}$, where γ is some constant, are satisfied everywhere in G.

Lemma 5.2.1. *Let* $S(\delta, z) \subset G$ *and let* $\mu_q(t)$ *be the function equal to* $2\sqrt{\delta^2 - (t - q(z))^2 |\operatorname{grad}[q(z)]|^{-2}}$ *on*

$$q(z) - \delta|\operatorname{grad}[q(z)]| \leqslant t \leqslant q(z) + \delta|\operatorname{grad}[q(z)]|$$

and equal to zero elsewhere. Then

$$\int_{-\infty}^{\infty} |\mu_q(t) - h_1(e(q, t) \cap S(\delta, z))|\, dt \leqslant c_1(\gamma)\, \omega(\delta)\, \delta^2,$$

where $c_1(\gamma)$ *is a constant depending only on* γ.

Proof. Let $[a, b] \subset e(q, t) \cap S(\delta, z)$ be the segment of the level curve $e(q, t)$, endpoints a and b, lying on the boundary of $S(\delta, z)$; $[z, a]$ and $[z, b]$ the vectors with origin at z and endpoints at a and b, respectively;

$$\alpha_1 = \gamma(\overrightarrow{[z, a]}, \operatorname{grad}[q(z)]), \quad \alpha_2 = \gamma(\overrightarrow{[z, b]}, \operatorname{grad}[q(z)]).$$

We have

$$|t - q(z)| = |q(a) - q(z)| = \left| \int_{s \in [z, a]} \frac{\partial q}{\partial s}\, ds \right|$$
$$= \delta \cos \alpha_1 |\operatorname{grad}[q(z)]| (1 + 0(1)\, \omega(\delta))$$

Hence

$$\delta \sin \alpha_1 = \sqrt{\delta^2 - (t - q(z) + 0(\gamma)\,\delta\omega(\delta))^2 \,|\, \mathrm{grad}\ [q(z)] \,|^{-2}}$$

and similarly

$$\delta \sin \alpha_2 = \sqrt{\delta^2 - (t - q(z) + 0(\gamma)\,\delta\omega(\delta))^2 \,|\, \mathrm{grad}\ [q(z)] \,|^{-2}}$$

By b) the size of the angle swept out by the tangent vector to the level curve $e(q, t)$ on moving along $[a, b]$ does not exceed $C_2(\gamma)\,\omega(\delta)$. Therefore

$$h_1([a, b]) = \delta(\sin \alpha_1 + \sin \alpha_2)(1 + 0(\gamma)\,\omega(\delta))$$
$$= 2\sqrt{\delta^2 - (t - q(z) + 0(\gamma)\,\delta\omega(\delta))^2 \,|\, \mathrm{grad}\ [q(z)] \,|^{-2}} + 0(\gamma)\,\delta\omega(\delta).$$

If $\alpha_1 \geqslant C_3(\gamma)\,\omega(\delta)$ (C_3 is a sufficiently large constant), then $[a, b] = e(q, t) \cap S(\delta, z)$. Consequently, for

$$|\, t - q(z) \,| \leqslant \theta = \delta \cos [C_3\,\omega(\delta)] \,|\, \mathrm{grad}\ [q(z)] \,| \times (1 + 0(1)\,\omega(\delta))$$

we have $h_1(e(q, t) \cap S(\delta, z)) = h_1([a, b])$. Since for every t (by b))

$$h_1(e(q, t) \cap S(\delta, z)) \leqslant C_4(\gamma)\,\delta\,(1 + \omega(\delta)),$$

we have

$$\int_{-\infty}^{\infty} |\, h_1(e(q, t) \cap S(\delta, z)) - \mu_q(t) \,|\, dt =$$
$$= \int_{q(z) - \Theta}^{q(z) + \Theta} |\, h_1(e(q, t) \cap S(\delta, z)) - \mu_q(t) \,|\, dt + 0(\gamma)\,\delta^2\omega(\delta).$$

We now estimate

$$\int_{q(z) - \Theta}^{q(z) + \Theta} |\, h_1(e(q, t) \cap S(\delta, z)) - \mu_q(t) \,|\, dt =$$
$$= \int_{q(z) - \Theta}^{q(z) + \Theta} |\, h_1([a, b]) - \mu_q(t) \,|\, dt \leqslant$$
$$\leqslant 2 \int_{q(z) - \Theta}^{q(z) + \Theta} \Big(\sqrt{\delta^2 - (t - q(z) + 0(\gamma)\,\delta\omega(\delta))^2 \,|\, \mathrm{grad}\ [q(z)] \,|^{-2}}$$
$$- \sqrt{\delta^2 - (t - q(z))^2 \,|\, \mathrm{grad}\ [q(z)] \,|^{-2}}\Big)\, dt + 0(\gamma)\,\delta^2\omega(\delta)$$
$$= 0(\gamma)\,\delta^2\omega(\delta) \int_{-1}^{1} \frac{d\tau}{\sqrt{1 - \tau^2}} + 0(\gamma)\,\delta^2\omega(\delta) = 0(\gamma)\,\delta^2\omega(\delta).$$

Here we have the mean value theorem. This proves the lemma.

LEMMA 5.2.2. *Let* $p(z), q(z)$ *satisfy conditions a) and b);* $S(\delta, z) \subset G$; *let* $f(t)$ *be an arbitrary continuous function, uniformly bounded in modulus by the constant* m. *Then*

$$\iint_{(u,v)\in S(\delta,z)} p(u,v) f(q(u,v))\, du dv$$

$$= p(z)\,|\operatorname{grad}[q(z)]|^{-1} \int_{-\infty}^{\infty} f(t)\mu_q(t)\,dt + \lambda(z)\, m\delta^2\omega(\delta),$$

where $|\lambda(z)| \leqslant C_5(\gamma)$.

Proof. Using a) and b) and Lemma 5.2.1 we have

$$\int_{S(\delta,z)} p(u,v) f(q(u,v))\, du dv$$

$$= p(z) \iint_{(u,v)\in S(\delta,z)} f(q(u,v))\, du dv + 0(1)\, m\delta^2\omega(\delta)$$

$$= p(z) \int_{-\infty}^{\infty} \{f(t) \int_{s\in e(q,t)\cap S(\delta,z)} |\operatorname{grad}[q(s)]|^{-2} ds\}\, dt + 0(1)\, m\delta^2\omega(\delta)$$

$$= p(z)\,|\operatorname{grad}[q(z)]|^{-1} \int_{-\infty}^{\infty} \{f(t) \int_{s\in e(q,t)\cap S(\delta,z)} ds\}\, dt + 0(\gamma)\, m\delta^2\omega(\delta)$$

$$= p(z)\,|\operatorname{grad}[q(z)]|^{-2} \int_{-\infty}^{\infty} f(t)\, h_1(e(q,t)\cap S(\delta,z))\, dt + 0(\gamma)\, m\delta^2\omega(\delta)$$

$$= p(z)\,|\operatorname{grad}[q(z)]|^{-1} \int_{-\infty}^{\infty} f(t)\mu_q(t)\, dt + 0(\gamma)\, m\delta^2\omega(\delta).$$

This proves the lemma.

LEMMA 5.2.3. *Suppose that a number* $\alpha > 0$ *and functions* $p(z)$, $q(z), f(t)$ *satisfying the conditions of Lemma* 5.2.2. *are given. If for every integer* k *such that*

$$\min_{z\in G} q(z) \leqslant t_k = k\delta\frac{\alpha}{m} \leqslant \max_{z\in G} q(z)$$

and any integer l *such that*

$$\min_{z\in G} |\operatorname{grad}[q(z)]| \leqslant t_l' = l\frac{\alpha}{m} \leqslant \max_{z\in G} |\operatorname{grad}[q(z)]|,$$

the inequality

$$\left| \int_{t_k - t_l'\delta}^{t_k + t_l'\delta} f(t) \sqrt{\delta^2 - \left(\frac{t - t_k}{t_l'}\right)^2}\, dt \right| \leqslant \alpha\delta^2$$

is satisfied, then for every disc $S(\delta, z) \subset G$

$$\Big| \iint_{(u,v)\in S(\delta,z)} p(u,v) f(q(u,v))\, du dv \Big| \leqslant c_6(\gamma)\left(\alpha\delta^2 + m\delta^2\omega(\delta)\right).$$

Proof. Suppose that a disc $S(\delta, z) \subset G$ is given. By the condition of the lemma there are integers k and l such that $|q(z) - t_k| \leqslant \delta\alpha/m$ and $\big|\,|\operatorname{grad}[q(z)]| - t_l'\big| \leqslant \alpha/m$. From Lemma 5.2.2 we obtain

$$\Big| \int_{(u,v)\in S(\delta,z)} p(u,v) f(q(u,v))\, du dv \Big| \leqslant \frac{|p(z)|}{|\operatorname{grad}[q(z)]|} \Big| \int_{-\infty}^{\infty} f(t)\mu_q(t)\, dt \Big|$$

$$+ c_5(\gamma)\, m\delta^2\omega(\delta) \leqslant \frac{2}{\gamma^2}\Bigg| \int_{q(z) - \delta|\operatorname{grad}[q(z)]|}^{q(z) + \delta|\operatorname{grad}[q(z)]|} f(t)\sqrt{\delta^2 - \frac{(t-q(z))^2}{|\operatorname{grad}[q(z)]|^2}}\, dt$$

$$- \int_{t_k - t_l'\delta}^{t_k + t_l'\delta} f(t)\sqrt{\delta^2 - \left(\frac{t - t_k}{t_l'}\right)^2}\, dt \Bigg| + \frac{2}{\gamma^2}\alpha\delta^2 + c_5(\gamma)\, m\delta^2\omega(\delta) \leqslant$$

(by the mean value theorem)

$$\leqslant \frac{2}{\gamma^2}\alpha\delta^2 + c_5(\gamma)\, m\delta^2\omega(\delta) + \frac{2}{\gamma^2}\left(\int_{-1}^{1} \frac{\delta m d\tau}{\sqrt{1-\tau^2}}\right)\delta\frac{\alpha}{m}$$

$$+ \frac{2}{\gamma^2}\left(\int_{-1}^{1} \frac{\delta^2 m d\tau}{\sqrt{1-\tau^2}}\right)\frac{\alpha}{m} \leqslant c_6(\gamma)\left(\alpha\delta^2 + m\delta^2\omega(\delta)\right).$$

This proves the lemma.

We denote by $F_m = F_m(D; p_1, p_2, \ldots, p_N; q_1, q_2, \ldots, q_N)$ the set of superpositions of the form

$$f(x,y) = \sum_{i=1}^{N} p_i(x,y) f_i(q_i(x,y)), \text{ where } \{p_i(x,y)\}$$

and $\{q_i(x,y)\}$ are fixed functions, defined in the closed region D of the x, y plane and satisfying conditions a) and b) with a constant γ not depending on i and $\{f_i(t)\}$ are arbitrary continuous functions, defined on $\{[a_i, b_i]\} = \{[\min_{z\in D} q_i(z); \max_{z\in D} q_i(z)]\}$ and uniformly bounded in modulus by the constant m.

THEOREM 5.2.1. *There exist constants* A *and* B *such that if* $\varepsilon > Am\omega(\delta)$ *then for the* (ε, δ)*-entropy of the set of functions* F_m, $H_{\varepsilon,\delta}(F_m) \leqslant \frac{B}{\delta}\left(\frac{m}{\varepsilon}\right)^2$, *where* A *and* B *depend only on* γ, N *and* D.

Proof. We put

$$R(f(z), \delta) = \max_{S(\delta,z) \subset D} \left| \frac{1}{\pi\delta^2} \iint_{(u,v) \in S(\delta,z)} f(u,v)\, du dv \right| .$$

We denote by $\mathscr{H}_{\varepsilon,\delta}(F_m)$ the ε-entropy of the space F_m, taking as the distance between the functions $f_1(z), f_2(z) \in F_m$ the number $R(f_1(z) - f_2(z), \delta)$. The inequality $H_{2\varepsilon,\delta}(F_m) \leqslant \mathscr{H}_{\varepsilon,\delta}(F_m)$ holds owing to the fact that if two functions $f_1(z)$ and $f_2(z)$ are (ε, δ)-distinguishable, then they are ε-distinguishable also in the sense of the metric $R(f_1(z) - f_2(z), \delta)$. We now estimate the value of $\mathscr{H}_{\varepsilon,\delta}(F_m)$. Let k and l be integers such that

$$\min_{z \in D} q_i(z) \leqslant t_k = k\delta \frac{\alpha}{m} \leqslant \max_{z \in D} q_i(z)$$

and

$$\min_{z \in D} |\operatorname{grad}[q_i(z)]| \leqslant t_l' = l \frac{\alpha}{m} \leqslant \max_{z \in D} |\operatorname{grad}[q_i(z)]| .$$

To compute the function

$$f_\delta(z) = \frac{1}{\pi\delta^2} \iint_{(u,v) \in S(\delta,z)} f(u,v)\, du dv ,$$

where $f(x, y) \in F_m$, $S(\delta, z) \subset D$ to within ε, it is sufficient by Lemma 5.2.3 to give the values of

$$v_i(t_k, t_l') = \frac{1}{\pi\delta^2} \int_{t_k - t_l'\delta}^{t_k + t_l'\delta} f_i(t) \sqrt{\delta^2 - \left(\frac{t - t_k}{t_l'}\right)^2}\, dt$$

to within $\alpha = \pi\varepsilon / (2\, NC_B(\gamma))$ and to assume that δ is small enough so that

$$\varepsilon > \frac{2NC_B(\gamma)\, m\omega(\delta)}{\pi} = A(\gamma, N)\, m\omega(\delta) .$$

Since $|v_i(t_k, t_l')| \leqslant C_1 m$, to write the numbers $v_i(t_k, t_l')$ (i, k, l fixed) $\log_2 (C_1 m/\alpha)$ binary digits are sufficient. Since

$$\left| v_i(t_{k+1}, t_l') - v_i(t_k, t_l') \right| \leqslant c_8 \frac{1}{\delta^2} \left(\int_{-1}^{1} \frac{\delta m d\tau}{\sqrt{1-\tau^2}} \right) \delta \frac{\alpha}{m} = c_9(\gamma)\alpha$$

(here we again use the mean value theorem), to store the numbers $v_i(t_{k+1}, t_l') - v_i(t_k, t_l')$ to within α, $\log_2 C_9$ binary digits are sufficient. Therefore to write the numbers $v_i(t_k, t_l')$ (i, l fixed; k any admissible number) $C_{10}(\gamma)\left[\log_2 \frac{m}{\alpha} + (b_i - a_i)\frac{m}{\delta\alpha}\right] = \mathscr{H}_{i,l}$ binary digits are sufficient. Consequently the total number of digits sufficient to store all the numbers $v_i(t_k, t_l')$ to within α, that is, to store the functions $f_\delta(z)$ to within ε, is

$$\mathscr{H} = \sum_{i,l} \mathscr{H}_{i,l} \leqslant Nc_{10}(\gamma)\left[\log_2 \frac{m}{\alpha} + (b_i - a_i)\frac{m}{\delta\alpha}\right]\frac{1}{\gamma}\frac{m}{\alpha} \leqslant \frac{B(\gamma, N, D)}{\delta}\left(\frac{m}{\varepsilon}\right)^2.$$

This proves the theorem.

§ 3. *Functional "dimension" of the space of linear superpositions*

Suppose that continuous functions $p_i(x, y)$ and continuously differentiable functions $q_i(x, y)$ ($i = 1, 2, \ldots, N$) are fixed. Let G be a closed region of the x, y plane. We denote by $F = F(G, \{p_i\}, \{q_i\})$ the set of superpositions of the form $f(x, y) = \sum_{i=1}^{N} p_i(x, y) f_i(q_i(x, y))$, where $(x, y) \in G$ and $\{f_i(t)\}$ are arbitrary continuous functions of one variable. We are interested in the functional dimension of the set F.

THEOREM 5.3.1. *In every region* D *of the* x, y *plane there exists a closed subregion* $G \subset D$ *such that*

$$r(F(G, \{p_i\}, \{q_i\})) \leqslant 1.$$

Proof. By Theorem 4.5.1, in D there exists a closed subregion $G^* \subset D$ such that the set of superpositions $F(G^*, \{p_i\}, \{q_i\})$ is closed (in the uniform metric) in $C(G^*)$, and the functions $\{q_i(x, y)\}$ satisfy the condition: for any i, either grad $[q_i(x, y)] \neq 0$ on G^* or $q_i(x, y) \equiv$ const on G^*. We show that $r(F(G^*, \{p_i\}, \{q_i\})) \leqslant 1$. By Banach's open mapping theorem, there exists a constant K such that for any superposition $\sum_{i=1}^{N} p_i(x, y) f_i(q_i(x, y)) = f(x, y) \in F(G^*, \{p_i\}, \{q_i\})$ there are con-