

# §4. Variation of superpositions of smooth functions

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tinuous functions  $\{f_i^*(t)\}$ , defined on the sets  $\{q_i(G^*)\}$  and satisfying the conditions

$$8) \quad f(x, y) = \sum_{i=1}^N p_i(x, y) f_i^*(q_i(x, y)) \text{ for all } (x, y) \in G^* ;$$

$$9) \quad \max_i \max_{t \in q_i(G^*)} |f_i^*(t)| \geq K \max_{(x, y) \in G^*} |f(x, y)| .$$

Denote by  $F_{\lambda\varepsilon} = F_{\lambda\varepsilon}(G^*, \{p_i\}, \{q_i\})$  the set of superpositions  $f(x, y) \in F(G^*, \{p_i\}, \{q_i\})$  such that  $\max_{(x, y) \in G^*} |f(x, y)| \leq \lambda\varepsilon$ . By Theorem 5.2.1

and (8), (9), there exist constants  $A$  and  $B$  such that if  $\omega(\delta) \leq (\lambda AK)^{-1}$  then  $H_{\varepsilon, \delta}(F_{\lambda\varepsilon}) \leq B(\lambda K)^2/\delta$ . Hence the functional dimension

$$r(F_i(G^*, \{p_i\}, \{q_i\})) \leq \lim_{\lambda \rightarrow \infty} \lim_{\delta \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \frac{\log_2 \log_2 \frac{B(\lambda K)^2}{\delta}}{\log_2 \delta} = 1$$

This proves the theorem.

From Theorem 5.3.1 and the properties of functional dimension (§ 1) we have the following result, which is a stronger form of Theorem 4.6.1.

**COROLLARY 5.3.1.** *For any continuous functions  $\{p_i(x, y)\}$  and continuously differentiable functions  $\{q_i(x, y)\}$  and every region  $D$  the set of linear superpositions  $F(D, \{p_i\}, \{q_i\})$  is nowhere dense in any space of functions that has in every region  $G \subset D$  functional "dimension" greater than 1.*

*Remark 5.3.1.* All the results about linear superpositions of the form  $\sum_{i=1}^N p_i(x, y) f_i(q_i(x, y))$  remain valid if we assume that  $\{f_i(t)\}$  are arbitrary bounded measurable functions.

#### § 4. Variation of superpositions of smooth functions

Let  $G_n$  be a closed region of the space of the variables  $x_1, x_2, \dots, x_n$  ( $n \geq 2$ ). A function  $F(x) = F(x_1, x_2, \dots, x_n)$  is called a superposition of order  $s$  generated by the functions of  $k$  ( $k > 1$ ) variables

$$f_{\beta_1, \beta_2, \dots, \beta_\alpha}(t_1, t_2, \dots, t_k) \quad (\alpha = 0, 1, 2, \dots, s; \beta_i = 1, 2, \dots, k)$$

if it is defined in  $G$  by relations

$$\left\{ \begin{array}{l} F = f(q_1, q_2, \dots, q_k), \\ \dots \\ q_{\beta_1, \beta_2, \dots, \beta_\alpha} = f_{\beta_1, \dots, \beta_\alpha}(q_{\beta_1, \dots, \beta_\alpha, 1}, q_{\beta_1, \dots, \beta_\alpha, 2}, \dots, q_{\beta_1, \dots, \beta_\alpha, k}), \\ \dots \\ q_{\beta_1, \beta_2, \dots, \beta_{s+1}} = x_{\gamma(\beta_1, \beta_2, \dots, \beta_{s+1})}, \end{array} \right. \quad (VI)$$

where  $\gamma(\beta_1, \beta_2, \dots, \beta_{s+1})$  is a function of the indices  $\beta_1, \beta_2, \dots, \beta_{s+1}$  and takes one of the values  $1, 2, \dots, n$ . As before, we assume that the functions  $\{\varphi_{\beta_1, \beta_2, \dots, \beta_\alpha}(t_1, t_2, \dots, t_k)\}$  are defined for all values of the arguments.

A superposition of any order, generated by functions of one variable, is again a function of one variable. Therefore in this case ( $k=1$ ) we consider superpositions of functions of one variable and the operation of addition, that is, superpositions definable in the following way.

A function  $F(x) = F(x_1, x_2, \dots, x_n)$  ( $n > 1$ ) is called a superposition of order  $s$  of the functions  $f_{\beta_1, \dots, \beta_\alpha}(t)$  ( $\alpha = 0, 1, 2, \dots, s$ ;  $\beta_i = 1, 2$ ) if the following relations are satisfied:

$$\left. \begin{array}{l} F = f(q_1 + q_2), \\ \dots \\ q_{\beta_1, \beta_2, \dots, \beta_\alpha} = f_{\beta_1, \beta_2, \dots, \beta_\alpha}(q_{\beta_1, \beta_2, \dots, \beta_\alpha, 1} + q_{\beta_1, \beta_2, \dots, \beta_\alpha, 2}), \\ \dots \\ q_{\beta_1, \beta_2, \dots, \beta_{s+1}} = x_{\gamma(\beta_1, \beta_2, \dots, \beta_{s+1})}, \end{array} \right\} \quad (VII)$$

where  $\gamma(\beta_1, \beta_2, \dots, \beta_{s+1})$  takes one of the values  $1, 2, \dots, n$ .

Note that we can represent as superpositions of the form (VII), for example, all rational functions of  $x_1, x_2, \dots, x_n$  since we can write any arithmetic operation by such superpositions, for example,  $u \cdot v = e^{\ln u + \ln v} = f(f_1(u) + f_2(v))$ .

Let  $F(x_1, x_2, \dots, x_n)$  be a superposition of order  $s$  of the continuously differentiable functions  $\{f_{\beta_1, \beta_2, \dots, \beta_\alpha}(t_1, t_2, \dots, t_k)\}$  and  $\tilde{F}(x_1, x_2, \dots, x_n)$  the superposition of the same form of the continuously differentiable functions  $\{\tilde{f}_{\beta_1, \beta_2, \dots, \beta_\alpha}(t_1, t_2, \dots, t_k)\}$ . We put

$$\begin{aligned} \varphi_{\beta_1, \beta_2, \dots, \beta_\alpha} &= \tilde{f}_{\beta_1, \dots, \beta_\alpha} - f_{\beta_1, \dots, \beta_\alpha} \quad (\alpha = 0, 1, 2, \dots, s; \beta_i = 1, 2, \dots, k) \\ \mu &= \max_{\alpha, \beta_1, \dots, \beta_\alpha} \sum_{i=1}^k \sup_t \left| \frac{\partial f_{\beta_1, \dots, \beta_\alpha}(t_1, \dots, t_k)}{\partial t_i} \right|, \\ \varepsilon &= \max_{\alpha, \beta_1, \dots, \beta_\alpha} \sup_t \left| \varphi_{\beta_1, \dots, \beta_\alpha}(t_1, t_2, \dots, t_k) \right| \end{aligned}$$

LEMMA 5.4.1. *The inequality*

$$\sup_{x \in G} \left| \tilde{F}(x_1, x_2, \dots, x_n) - F(x_1, x_2, \dots, x_n) \right| \leq A(\mu, s) \varepsilon.$$

holds, where the constant  $A(\mu, s)$  depends only on  $\mu$  and  $s$ .

*Proof.* We proceed by induction on  $s$ . For definiteness suppose that  $k < 1$ . Having verified the statement of the lemma for  $s = 1$  and having made an appropriate inductive assumption for superpositions of order  $s - 1$ , we have

$$\begin{aligned} \sup_{x \in G} \left| \tilde{F}(x_1, x_2, \dots, x_n) - F(x_1, x_2, \dots, x_n) \right| \\ \leq \left| f(\tilde{q}_1, \dots, \tilde{q}_k) - f(q_1, \dots, q_k) \right| + \left| \varphi(\tilde{q}_1, \tilde{q}_2, \dots, \tilde{q}_k) \right| \\ \leq \mu \max_{\beta_1} \sup_{x \in G} \left| \tilde{q}_{\beta_1} - q_{\beta_1} \right| + \varepsilon \leq \mu \cdot A(\mu, s-1) \varepsilon + \varepsilon = A(\mu, s) \varepsilon. \end{aligned}$$

(the last by the inductive assumption). This proves the lemma.

Further, let  $\omega(\delta)$  be the common modulus of continuity of all the functions  $\left\{ \frac{\partial f_{\beta_1, \dots, \beta_\alpha}(t_1, \dots, t_k)}{\partial t_i} \right\}$  and, in addition, put

$$\varepsilon' = \max_{\alpha, \beta_1, \dots, \beta_\alpha} \sum_{i=1}^k \sup_t \left| \frac{\partial \varphi_{\beta_1, \dots, \beta_\alpha}(t_1, \dots, t_k)}{\partial t_i} \right|$$

LEMMA 5.4.2. *We have (for case  $k > 1$ )*

$$\begin{aligned} \tilde{F}(x_1, \dots, x_n) - F(x_1, \dots, x_n) = \sum_{\alpha, \beta_1, \dots, \beta_\alpha} p_{\beta_1, \dots, \beta_\alpha}(x_1, x_2, \dots, x_n) \\ \times \varphi_{\beta_1, \dots, \beta_\alpha}(q_{\beta_1, \dots, \beta_\alpha, 1}(x_1, \dots, x_n), \dots, q_{\beta_1, \dots, \beta_\alpha, k}(x_1, \dots, x_n)) \\ + R(x_1, x_2, \dots, x_n), \end{aligned}$$

where

$$\left| R(x_1, x_2, \dots, x_n) \right| \leq B(\mu, s, k) [\varepsilon' + \omega(A(\mu, s) \varepsilon)] \varepsilon,$$

$$p_{\beta_1, \dots, \beta_\alpha}(x_1, x_2, \dots, x_n) = \prod_{i=0}^{\alpha-1} \frac{\partial f_{\beta_1, \dots, \beta_i}}{\partial q_{\beta_1, \dots, \beta_{i+1}}}$$

(for  $\alpha=0$   $p(x_1, x_2, \dots, x_n) \equiv 1$ ),

$B(\mu, s, k)$  is a constant depending only on  $\mu, s, k$ . For  $k = 1$  the corresponding equation is slightly different (see Chapter I, (III)):

$$\begin{aligned} & \tilde{F}(x_1, \dots, x_n) - F(x_1, \dots, x_n) \\ = & \sum_{\alpha, \beta_1, \dots, \beta_\alpha} p_{\beta_1, \dots, \beta_\alpha}(x_1, x_2, \dots, x_n) \varphi_{\beta_1, \dots, \beta_\alpha}(q_{\beta_1, \dots, \beta_\alpha, 1}(x_1, \dots, x_n) \\ & + q_{\beta_1, \dots, \beta_\alpha, 2}(x_1, \dots, x_n)) + R(x_1, \dots, x_n). \end{aligned}$$

*Proof.* As in the preceding lemma we proceed by induction on  $s$ . Again for definiteness we limit ourselves to the case  $k > 1$ . For  $s = 1$  the assertion of the lemma is easily verified. We assume that it is true for superpositions of order  $s - 1$ . By Lemma 5.4.1, for superpositions of order  $s$  we have

$$\begin{aligned} \tilde{F}(x_1, \dots, x_n) - F(x_1, \dots, x_n) &= f(\tilde{q}_1, \tilde{q}_2, \dots, \tilde{q}_k) - f(q_1, q_2, \dots, q_k) \\ &+ \varphi(\tilde{q}_1, \tilde{q}_2, \dots, \tilde{q}_k) - \varphi(q_1, q_2, \dots, q_k) + \sum_{\beta_1=1}^k \frac{\partial f}{\partial q_{\beta_1}}(\tilde{q}_{\beta_1} - q_{\beta_1}) \\ &+ A(\mu, s) \varepsilon' \cdot \varepsilon + k \cdot A(\mu, s) \omega(A(\mu, s) \varepsilon) \varepsilon. \end{aligned}$$

Since  $\tilde{q}_{\beta_1}$  and  $q_{\beta_1}$  ( $\beta_1 = 1, 2, \dots, k$ ) are superpositions of order  $s - 1$ , by the inductive hypothesis we have

$$\begin{aligned} \tilde{q}_{\beta_1} - q_{\beta_1} &= \sum_{\substack{\alpha > 0 \\ \beta_2, \beta_3, \dots, \beta_\alpha}} \hat{p}_{\beta_1, \dots, \beta_\alpha}(x_1, x_2, \dots, x_n) \\ &\times \varphi_{\beta_1, \dots, \beta_\alpha}(q_{\beta_1, \dots, \beta_\alpha, 1}(x_1, x_2, \dots, x_n), \dots, q_{\beta_1, \dots, \beta_\alpha, k}(x_1, x_2, \dots, x_n)) \\ &+ \hat{R}(x_1, x_2, \dots, x_n), \end{aligned}$$

where

$$\begin{aligned} |\hat{R}(x_1, x_2, \dots, x_n)| &\leq B(\mu, s - 1, k) [\varepsilon' + \omega(A(\mu, s - 1) \varepsilon)] \varepsilon, \\ \hat{p}_{\beta_1, \dots, \beta_\alpha}(x_1, \dots, x_n) &= \prod_{i=1}^{\alpha-1} \frac{\partial f_{\beta_1, \beta_2, \dots, \beta_i}}{\partial q_{\beta_1, \dots, \beta_{i+1}}} \end{aligned}$$

(for  $\alpha = 1, \hat{p}_{\beta_1}(x_1, \dots, x_n) \equiv 1$ ).

When we now substitute the expressions for the differences  $\tilde{q}_{\beta_1} - q_{\beta_1}$  in the formula for  $\tilde{F} - F$  above, we obtain the required representation of the difference of two superpositions  $\tilde{F} - F$ . This proves the lemma.