§5. Instability of the representation of functions as superpositions of smooth functions

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **23 (1977)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **30.06.2024**

§ 5. *Instability of the representation of functions as superpositions of smooth functions*

Let A be a set of functions of n variables and B a set of functions of k variables ($k<n$). Suppose that a function $F(x_1, \ldots, x_n) \in A$ is in a region G_n of the space $x_1, x_2, \ldots, x_n$ an s-fold superposition, generated by a system of functions $\{f_{\beta_1,\ldots,\beta_\alpha}(t_1, \ldots, t_k)\}$ of B.

We say that this superposition is (A, B)-stable in G_n if every function $\tilde{F}(x_1, \ldots, x_n) \in A$ can be represented in G_n as the s-fold superposition of the same form of functions $\{\tilde{f}_{\beta_1,\ldots,\beta_\alpha}(t_1, t_2, \ldots, t_k)\}$ of B such that

$$\max_{\alpha;\beta_1,\ldots,\beta_\alpha} \sup_t \left|\tilde{f}_{\beta_1,\ldots,\beta_\alpha}(t_1, \ldots, t_k) - f_{\beta_1,\ldots,\beta_\alpha}(t_1, \ldots, t_k)\right|$$
$$\leqslant \lambda \sup_{x \in G_n} \left|\tilde{F}(x_1, \ldots, x_n) - F(x_1, \ldots, x_n)\right|,$$

where λ is a constant not depending either on $\tilde{F}$ or on the $\{\tilde{f}_{\beta_1,\ldots,\beta_\alpha}\}$.

We denote by $C^{(1)}_{\omega(\delta)}$ the space of all continuously differentiable functions of k variables whose partial derivatives have modulus of continuity $\omega(\delta)$ ($\omega(\delta) \to 0$ as $\delta \to 0$).

THEOREM 5.5.1. *Suppose that each function* $F(x_1, \ldots, x_n) \in A$ *is in some region* D_n *of the space* $x_1, \ldots, x_n$ *a superposition of order* s *of functions of* k *variables* $\{f_{\beta_1,\ldots,\beta_\alpha}(t_1, \ldots, t_k)\}$ *belonging to* $C^{(1)}_{\omega(\delta)}$ $(k<n)$. *If for any subregion* $G_n \subset D_n$ *the functional "dimension" of* A *at* $F(x_1, \ldots, x_n) \in A$ *is greater than* k, *then the function* $F(x_1, \ldots, x_n)$ *cannot be an* $(A, C^{(1)}_{\omega(\delta)})$-*stable superposition in any such region* $G \subset D_n$.

Proof. Assume the contrary, that is, in a region $G_n \subset D_n$ the function $F(x_1, \ldots, x_n) \in A$ is an $(A, C^{(1)}_{\omega(\delta)})$-stable s-fold superposition of functions $\{f_{\beta_1,\ldots,\beta_\alpha}(t_1, \ldots, t_k)\}$ of $C^{(1)}_{\omega(\delta)}$. Then any function $\tilde{F}(x_1, \ldots, x_n) \in A$ can be represented as the superposition of the same form of functions $\{\tilde{f}_{\beta_1,\ldots,\beta_\alpha}(t_1, \ldots, t_k)\}$ of $C^{(1)}_{\omega(\delta)}$ such that

$$\max_{\alpha;\beta_1,\ldots,\beta_\alpha} \sup_t \left|\varphi_{\beta_1,\ldots,\beta_\alpha}(t_1, \ldots, t_k)\right| \leqslant \lambda \sup_{x \in G_n} \left|\tilde{F} - F\right|,$$

where $\varphi_{\beta_1,\ldots,\beta_\alpha} = \tilde{f}_{\beta_1,\ldots,\beta_\alpha} - f_{\beta_1,\ldots,\beta_\alpha}$. By Lemma 5.4.2 we have (for definiteness, $k > 1$)

$$\tilde{F} - F = \sum_{\alpha;\, \beta_1, \ldots, \beta_\alpha} p_{\beta_1, \ldots, \beta_\alpha}(x_1, \ldots, x_n)$$

$$\times\ \varphi_{\beta_1, \ldots, \beta_\alpha}(q_{\beta_1, \ldots, \beta_\alpha, 1}(x_1, \ldots, x_n), \ldots, q_{\beta_1, \ldots, \beta_\alpha, k}(x_1, \ldots, x_n)) + R(x_1, \ldots, x_n),$$

where $| R(x_1, \ldots, x_n) | \leqslant \gamma(\varepsilon)\,\varepsilon$, $\gamma(\varepsilon) \to 0$ as $\varepsilon \to 0$, and

$$\varepsilon = \max_{\alpha;\, \beta_1, \ldots, \beta_\alpha} \sup_t | \varphi_{\beta_1, \ldots, \beta_\alpha}(t_1, \ldots, t_k) |$$

$$\leqslant \lambda \sup_{x \in G_n} | \tilde{F}(x_1, \ldots, x_n) - F(x_1, \ldots, x_n) | .$$

That $\gamma(\varepsilon) \to 0$ as $\varepsilon \to 0$ follows from the fact that as $\varepsilon \to 0$ the quantity

$$\varepsilon' = \max_{\alpha;\, \beta_1, \ldots, \beta_\alpha} \sum_{i=1}^{k} \sup \left| \frac{\partial \varphi_{\beta_1, \ldots, \beta_\alpha}(t_1, \ldots, t_k)}{\partial t_i} \right| \to 0,$$

provided only that the modulus of continuity of the partial derivatives of the functions $\{ \varphi_{\beta_1, \ldots, \beta_\alpha}(t_1, \ldots, t_k) \}$ is fixed. By 5.1.10 it follows that $r(A, F) \leqslant k$ in some subregion $G_n \subset D_n$. So we have obtained a contradiction to the assumption that $r(A, F) > k$ in any subregion $G_n \subset D_n$ and this proves the theorem.

REFERENCES

[1] Hilbert, D. Mathematische Probleme. *Nachr. Akad. Wiss. Göttingen* (1900), 253-297; *Gesammelte Abhandlungen*, Bd. 3 (1935), 290-329.

[2] Ostrowski, A. Über Dirichletsche Reihen und algebraische Differentialgleichungen. *Math. Z. 8* (1920), 241-298.

[3] Hilbert, D. Über die Gleichung neunten Grades. *Math. Ann. 97* (1927), 243-250; *Gesammelte Abhandlungen*, Bd. 2 (1933), 393-400.

[4] Vitushkin, A. G. On Hilbert's thirteenth problem. *Dokl. Akad. Nauk SSSR 95* (1954), 701-704.

[5] Bieberbach, L. Bemerkung zum dreizehnten Hilbertschen Problem. *J. Reine Angew. Math. 165* (1931), 89-92.

[6] —— Einfluss von Hilberts Pariser Vortrag über „Mathematische Probleme". *Naturwissenschaften 51* (1930), 1101-1111.

[7] Kolmogorov, A. N. On the representation of continuous functions of several variables by superpositions of continuous functions of fewer variables. *Dokl. Akad. Nauk SSSR 108* (1956), 179-182. *Amer. Math. Soc. Transl.* (2) 17 (1961), 369-373.

[8] Arnol'd, V. I. On functions of three variables. *Dokl. Akad. Nauk SSSR 114* (1957), 679-681.

[9] Kolmogorov, A. N. On the representation of continuous functions of several variables by superpositions of continuous functions of one variable and addition. *Dokl. Akad. Nauk SSSR 114* (1957), 953-956. *Amer. Math. Soc. Transl. (2) 28* (1963), 55-59.