

# 4. Derivation of Siegel's Theorem

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$e_i = 1 / \# E(S_i)$ , where  $\#$  denotes cardinality. We now define the “number of representations of  $T$  by the genus of  $S$ ” as

$$A(\text{genus}(S), T) = \frac{e_1 A(S_1, T) + \dots + e_h A(S_h, T)}{e_1 + \dots + e_h}.$$

Now  $S$  is a real symmetric matrix, and so we may view it as a point in  $\mathbf{R}^{n_1}$ , where  $n_1 = n(n+1)/2$ . Similarly,  $T$  is a point in  $\mathbf{R}^{m_1}$ . Let  $dt$  be the usual measure in  $\mathbf{R}^{m_1}$ , and let  $dx$  be the usual measure in the real vector space of  $m \times n$  matrices. Given  $\varepsilon > 0$ , let  $B_\varepsilon$  denote the  $\varepsilon$ -neighborhood of  $T$  in  $\mathbf{R}^{m_1}$ , and let  $C_\varepsilon$  denote the set of  $x \in M_{m \times n}(\mathbf{R})$  satisfying  $S[x] \in B_\varepsilon$ . Then  $B_\varepsilon$  and  $C_\varepsilon$  are open sets with compact closure, and the following limit is known to exist:

$$A_\infty(S, T) = \lim_{\varepsilon \rightarrow 0} \int_{C_\varepsilon} dx / \int_{B_\varepsilon} dt.$$

THEOREM (Siegel [4]). For  $m - n \geq 3$ ,

$$(S) \quad A(\text{genus}(S), T) = A_\infty(S, T) \lim_q \frac{A_q(S, T)}{q^{mn - (n+1)n/2}}.$$

#### 4. DERIVATION OF SIEGEL'S THEOREM

Let  $G = \{g \in SL(m) : S[g] = S\}$ , and let  $X = \{x \in M_{m \times n} : S[x] = T\}$ . If  $m \geq 4$ , both  $G_{\mathbf{C}}$  and  $G_{\mathbf{R}}$  have fundamental groups of order 2. Condition (c) of § 2 is the classical Witt theorem for  $(G, X)$ . We assume that  $X_{\mathbf{Q}}$  is nonempty.

We will show that (A) implies (S). This reduces Siegel's theorem to the computation of the Tamagawa number  $\tau(G)$ .

Let  $\Phi_\infty =$  the constant function 1 on  $X_{\mathbf{R}}$ , and let  $\Phi_p =$  the characteristic function of  $X_{\mathbf{Z}_p}$  in  $X_{\mathbf{Q}_p}$ . Then  $\Phi = \Phi_\infty \cdot \prod \Phi_p$  is the characteristic function of  $X_{S_\infty} = X_{\mathbf{R}} \cdot \prod X_{\mathbf{Z}_p}$  in  $X_A$ . Because of the positive definiteness of  $S$ ,  $\Phi$  has compact support.

Consider the right-hand side of formula (S). Siegel has shown that there exists an algebraic gauge form  $dx$  on  $X$  such that  $A_\infty(S, T) = \int_{X_{\mathbf{R}}} dx_\infty$ , and

$$\lim_q \frac{A_q(S, T)}{q^{mn - (n+1)n/2}} = \prod_p \int_{X_{\mathbf{Z}_p}} dx_p,$$

where  $dx$  and  $dx_p$  are the positive measures induced on  $X_{\mathbf{R}}$  and  $X_{\mathbf{Q}_p}$  by  $dx$ .

It remains to identify the left-hand sides of (A) and (S). First we analyze the denominator of the left-hand side of (A). Since  $\tau(G) = \tau(G_\xi) = 2$  ([5]), this denominator is  $\int_{G_A/G_Q} dg$ . Now  $G_A$  admits a double coset decomposition

$$G_A = G_{S_\infty} \sigma_1 G_Q \dots G_{S_\infty} \sigma_h G_Q.$$

Then, following Tamagawa [5].

$$\begin{aligned} \int_{G_A/G_Q} dg &= \sum_1^h \int_{(G_{S_\infty} \sigma_i G_Q)/G_Q} dg = \\ &= \sum_1^h \int_{(\sigma_i^{-1} G_{S_\infty} \sigma_i G_Q)/G_Q} dg \\ &= \sum_1^h \int_{(\sigma_i^{-1} G_{S_\infty} \sigma_i)/G(\sigma_i)} dg, \end{aligned}$$

where  $G(\sigma_i) = \sigma_i^{-1} G_{S_\infty} \sigma_i \cap G_Q$ . This reduces to

$$\frac{\sum_1^h \int_{G_{S_\infty}} dg}{\# G(\sigma_i)} = \int_{G_{S_\infty}} dg \cdot \sum_1^h e_i.$$

A similar reduction applies to the numerator. First observe that for our choice of  $\Phi$ ,

$$\begin{aligned} \sum_{x \in X_Q} \Phi(gx) &= \#(X_Q \cap g^{-1} X_{S_\infty}) \\ &= \#(gX_Q \cap X_{S_\infty}). \end{aligned}$$

Then

$$\begin{aligned} \int_{G_A/G_Q} \sum_{x \in X_Q} \Phi(gx) dg &= \int_{G_A/G_Q} \#(gX_Q \cap X_{S_\infty}) dg \\ &= \sum_1^h \int_{(G_{S_\infty} \sigma_i G_Q)/G_Q} \#(gX_Q \cap X_{S_\infty}) dg = \\ &= \sum_1^h \int_{(\sigma_i^{-1} G_{S_\infty} \sigma_i G_Q)/G_Q} \#(\sigma_i gX_Q \cap X_{S_\infty}) dg \\ &= \sum_1^h \int_{(\sigma_i^{-1} G_{S_\infty} \sigma_i)/G(\sigma_i)} \#(\sigma_i gX_Q \cap X_{S_\infty}) dg \\ &= \sum_1^h e_i \int_{G_{S_\infty}} \#(g\sigma_i X_Q \cap X_{S_\infty}) dg \\ &= \sum_1^h e_i \int_{G_{S_\infty}} \#(\sigma_i X_Q \cap g^{-1} X_{S_\infty}) dg \end{aligned}$$

$$\begin{aligned}
 &= \sum_1^h e_i \int_{G_{S_\infty}} \# (\sigma_i X_{\mathbf{Q}} \cap X_{S_\infty}) dg \\
 &= \int_{G_{S_\infty}} dg \cdot \sum_1^h e_i \# (\sigma_i X_{\mathbf{Q}} \cap X_{S_\infty}).
 \end{aligned}$$

The left-hand side of (A) therefore becomes

$$\frac{\int_{G_{S_\infty}} dg \cdot \sum_1^h e_i \# (\sigma_i X_{\mathbf{Q}} \cap X_{S_\infty})}{\int_{G_{S_\infty}} dg \cdot \sum_1^h e_i} = \frac{\sum_1^h e_i \# (\sigma_i X_{\mathbf{Q}} \cap X_{S_\infty})}{\sum_1^h e_i}.$$

The following result completes the identification of the left-hand side of (A) with  $A$  (genus  $(S), T$ ).

PROPOSITION.  $A(S_i, T) = \# (\sigma_i X_{\mathbf{Q}} \cap X_{S_\infty})$ .

Before giving the proof, we reinterpret the matrices  $S_1, \dots, S_h$ .  $G_A$  acts on the set of  $\mathbf{Z}$ -lattices in  $\mathbf{Q}^n$  as follows: for  $\sigma \in G_A$  and a lattice  $L$ ,  $\sigma^* L$  is the unique lattice satisfying

$$(\sigma^* L) \otimes \mathbf{Z}_p = \sigma_p(L \otimes \mathbf{Z}_p),$$

for all  $p$ .

The matrix  $S$  defines a quadratic form on  $\mathbf{Q}^n$  by  $q(x) = S[x]$ . Consider the lattices  $\sigma_1^* \mathbf{Z}^n, \dots, \sigma_h^* \mathbf{Z}^n$ . In each lattice  $\sigma_i^* \mathbf{Z}^n$  choose a  $\mathbf{Z}$ -basis, and let  $S_i$  be the matrix of  $q$  with respect to this basis. Then  $S_1, \dots, S_h$  form a complete set of representatives of the  $h$  classes in genus  $(S)$  (see [7]).

Decomposing  $(SL_m)_A = (SL_m)_{S_\infty} (SL_m)_{\mathbf{Q}}$  we see that each  $\sigma_i \in G_A$  can be written  $\sigma_i = u_i a_i$ , where  $u_i \in (SL_m)_{S_\infty}$ ,  $a_i \in (SL_m)_{\mathbf{Q}}$ . Then

$$\begin{aligned}
 \sigma_i^{-1} * \mathbf{Z}^m &= a_i^{-1} u_i^{-1} * \mathbf{Z}^m = a_i^{-1} * (u_i^{-1} * \mathbf{Z}^m) \\
 &= a_i^{-1} * \mathbf{Z}^m = a_i^{-1} \mathbf{Z}^m.
 \end{aligned}$$

Let  $w_1, \dots, w_m$  be the standard  $\mathbf{Z}$ -basis of  $\mathbf{Z}^m$ ; then  $a_i^{-1} w_1, \dots, a_i^{-1} w_m$  is a  $\mathbf{Z}$ -basis of  $\sigma_i^{-1} * \mathbf{Z}^m$ . The matrix of  $q$  with respect to this basis is

$$S_i = S[a_i^{-1}] = S[\sigma_i][a_i^{-1}] = S[\sigma_i a_i^{-1}] = S[u_i].$$

LEMMA. Let  $X_i = \{x \in M_{m \times n} : S_i[x] = T\}$ . Then

- (1)  $(X_i)_{\mathbf{Q}} = a_i X_{\mathbf{Q}}$ ,
- (2)  $(X_i)_{S_\infty} = u_i^{-1} X_{S_\infty}$ .

*Proof of (1):* Let  $x \in X_A$ .  $a_i x \in a_i X_{\mathbf{Q}} \Leftrightarrow x$  is  $\mathbf{Q}$ -rational and  $S[x] = T \Leftrightarrow a_i x$  is  $\mathbf{Q}$ -rational and  $S_i[a_i x] = T \Leftrightarrow a_i x \in (X_i)_{\mathbf{Q}}$ .

The proof of (2) is similar.

Now we prove the proposition.

$$\begin{aligned} A(S_i, T) &= \#(X_i)_{\mathbf{Z}} = \#((X_i)_{\mathbf{Q}} \cap (X_i)_{S_{\infty}}) = \#(a_i X_{\mathbf{Q}} \cap u_i^{-1} X_{S_{\infty}}) \\ &= \#(u_i a_i X_{\mathbf{Q}} \cap X_{S_{\infty}}) = \#(\sigma_i X_{\mathbf{Q}} \cap X_{S_{\infty}}). \end{aligned}$$

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