

## 2. Proof of theorem B

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THEOREM B. Let  $m, \chi, f, \psi$  be as above. We put

$$q = \prod_{\substack{p|m \\ p \nmid f}} p, \quad R = \frac{m}{fq}$$

where  $p$  denotes rational primes. Let the multiplicative function  $g$  be defined by

$$g(n) = \mu((n, q)) \varphi((n, q)) \bar{\psi}(n).$$

Then we have:

$$\left. \begin{aligned} \mathcal{G}(\alpha, \chi) &= 0 \text{ if } R \nmid \alpha \\ \mathcal{G}(Rn, \chi) &= \mu(q) \psi(q) \tau(\psi) R g(n), \quad n = 1, 2, \dots \end{aligned} \right\} \quad (3)$$

## 2. PROOF OF THEOREM B

For a Dirichlet character  $\chi$  mod  $m$  let the function  $L(s, \chi)$  be given by

$$L(s, \chi) = \sum_1^{\infty} \chi(n) n^{-s}, \quad \text{Re } s > 1.$$

The series defines an analytic function for  $\text{Re } s > 1$ , which can be extended to a meromorphic function on the whole complex plane, with at most one simple pole at  $s = 1$ . If  $\chi$  is primitive, then  $L(s, \chi)$  satisfies the equation

$$L(1-s, \chi) = m^{s-1} (2\pi)^{-s} \Gamma(s) \left( e^{-\frac{\pi is}{2}} + \chi(-1) e^{\frac{\pi is}{2}} \right) \tau(\chi) L(s, \bar{\chi}). \quad (4)$$

Because of (1), this can also be written, for  $\text{Re } s > 1$ , as

$$L(1-s, \chi) = m^{s-1} (2\pi)^{-s} \Gamma(s) \left( e^{-\frac{\pi is}{2}} + \chi(-1) e^{\frac{\pi is}{2}} \right) \sum_1^{\infty} \mathcal{G}(n, \chi) n^{-s}. \quad (5)$$

Whereas (4) holds only for primitive characters, (5) turns out to be valid in the general case. In fact, a much more general formula is proved in [3], th. 6.1; if we put there  $x = \alpha = 0$ ,  $\alpha_n = \chi(n)$  and observe that  $\mathcal{G}(-n, \chi) = \chi(-1) \mathcal{G}(n, \chi)$ , we get (5) immediately. But also most of the classical (= non-adelic) proofs of (4) will give (5) after very small changes. The only use of the primitivity of  $\chi$  in these proofs is that they replace  $\mathcal{G}(n, \chi)$  by  $\bar{\chi}(n) \tau(\chi)$  at some stage (See for instance [7]).

Now let  $\chi, m, \psi, f$  be as in the theorem. We have, by the Euler-product,

$$\begin{aligned} L(1-s, \chi) &= L(1-s, \psi) \prod_{p|q} \left( 1 - \frac{\psi(p)}{p^{1-s}} \right) \\ &= L(1-s, \psi) \mu(q) q^{s-1} \psi(q) \prod_{p|q} (1 - p\bar{\psi}(p) p^{-s}) \end{aligned} \quad (6)$$

Formula (4) is valid if we write  $f$  for  $m$  and  $\psi$  for  $\chi$ . Eliminating  $L(1-s, \chi)$  and  $L(1-s, \psi)$  out of the equations (4), (5), (6), and taking into account that  $\chi(-1) = \psi(-1)$ , we obtain after an easy calculation, for  $\text{Re } s > 1$ :

$$\sum_1^{\infty} \mathcal{G}(n, \chi) n^{-s} = R \tau(\psi) \mu(q) \psi(q) R^{-s} L(s, \bar{\psi}) \prod_{p|q} \left(1 - \frac{p\bar{\psi}(p)}{p^s}\right) \quad (7)$$

Using Euler's product, we have for  $\text{Re } s > 1$

$$\begin{aligned} L(s, \bar{\psi}) \prod_{p|q} (1 - p\bar{\psi}(p) p^{-s}) \\ &= \prod_{p \nmid q} \sum_{k=0}^{\infty} \bar{\psi}(p^k) p^{-ks} \cdot \prod_{p|q} (1 + (1-p) \sum_{k=1}^{\infty} \bar{\psi}(p^k) p^{-ks}) \\ &= \prod_p \sum_{k=0}^{\infty} \bar{\psi}(p^k) \mu((p^k, q)) \varphi((p^k, q)) p^{-ks} \\ &= \prod_p \sum_{k=0}^{\infty} g(p^k) p^{-ks} = \sum_1^{\infty} g(n) n^{-s}. \end{aligned}$$

If we put that into (7) we get

$$\sum_1^{\infty} \mathcal{G}(n, \chi) n^{-s} = R \tau(\psi) \mu(q) \psi(q) \sum_1^{\infty} g(n) (Rn)^{-s},$$

and we obtain (3) by comparing the coefficients.

3. In this section we show that one can prove theorem  $A$  using theorem  $B$  and conversely. Let  $K(\alpha)$  resp.  $H(\alpha)$  be the right hand side of (2) resp. (3). We want to prove  $K(\alpha) = H(\alpha)$ . First we show that  $K(\alpha) \neq 0$  iff  $H(\alpha) \neq 0$ .

Now  $H(\alpha) \neq 0$  iff  $R|\alpha$  and  $(\alpha R^{-1}, f) = 1$ , and this is equivalent to

$$(\alpha f q, f m) = m \quad (8)$$

On the other hand  $K(\alpha) \neq 0$  iff the four conditions hold

$$(\alpha, m) f | m \quad (9)$$

$$\frac{m}{(\alpha, m) f} \text{ is squarefree} \quad (10)$$

$$\left(\frac{m}{(\alpha, m) f}, f\right) = 1 \quad (11)$$

$$\left(\frac{\alpha}{(\alpha, m)}, f\right) = 1 \quad (12)$$

At several places in the following we will use that  $q$  is squarefree,  $(f, q) = 1$  and the prime divisors of  $m$  are precisely the prime divisors of  $fq$ .

Let us assume (8). Then  $f(\alpha, m) \mid f(\alpha q, m) = m$ . Also  $\frac{m}{f(\alpha, m)} = \frac{(\alpha q, m)}{(\alpha, m)} \mid q$ . This proves (9) and (10). From  $f = m/(\alpha q, m)$  we have

$$(\alpha/(\alpha, m), f) = (\alpha/(\alpha, m), m/(\alpha q, m)) \mid (\alpha/(\alpha, m), m/(\alpha, m)) = 1.$$

This proves (12). Finally

$$\left( \frac{m}{(\alpha, m)f}, f \right) = \left( \frac{f(\alpha q, m)}{(\alpha, m)f}, f \right) \mid (q, f) = 1,$$

proving (11).

Conversely assume (9)-(12). From (10) and (11) we infer that  $\frac{m}{f(\alpha, m)} \mid q$ .

This implies

$$m \mid m \left( \frac{\alpha}{(\alpha, m)}, f \right) = \left( \frac{m\alpha f}{f(\alpha, m)}, mf \right) \mid (q\alpha f, mf). \quad (13)$$

Also

$$f(m, q\alpha) = f(\alpha, m) \left( \frac{m}{(\alpha, m)}, q \frac{\alpha}{(\alpha, m)} \right) = f(\alpha, m) \left( \frac{m}{(\alpha, m)}, q \right).$$

In the last term, the numbers  $f$  and  $(m/(\alpha, m), q)$  both divide  $m/(\alpha, m)$ , because of (9), and they are coprime, hence their product divides  $m/(\alpha, m)$ . This gives  $f(m, q\alpha) \mid m$ . Together with (13) this implies (8).

It remains to prove that  $H(\alpha) = K(\alpha)$  for  $\alpha = Rn$ ,  $(n, f) = 1$ . We have  $m_0 = m/(\alpha, m) = fq/(n, fq) = fq/(n, q)$ . Hence

$$\begin{aligned} R &= \frac{m}{fq} = \frac{\varphi(m)}{\varphi(fq)} = \frac{\varphi(m)}{\varphi(f)\varphi(q)} = \frac{\varphi(m)}{\varphi(f)\varphi((n, q))\varphi\left(\frac{q}{(n, q)}\right)} \\ &= \frac{\varphi(m)}{\varphi((n, q))\varphi(m_0)} \end{aligned}$$

hence

$$R\varphi((n, q)) = \varphi(m)/\varphi(m_0) \quad (14)$$

Also  $\mu(q) = \mu((n, q))\mu(q/(n, q))$ , so

$$\mu(q)\mu((n, q)) = \mu(q/(n, q)) = \mu(m_0/f). \quad (15)$$

Finally  $\alpha_0 = n/(n, q)$ , so  $\alpha_0 q = nm_0/f$ ,

hence

$$\psi(\alpha_0)\psi(q) = \psi(m_0/f)\psi(n)$$

or

$$\bar{\psi}(n)\psi(q) = \bar{\psi}(\alpha_0)\psi\left(\frac{m_0}{f}\right) \tag{16}$$

Multiplying (14), (15) and (16) we find  $K(\alpha) = H(\alpha)$ .

#### 4. SPECIAL CASES

(a) Theorem *B* implies that  $\tau(\chi) \neq 0$  if and only if  $R = 1$ , that is, if and only if  $m/f$  is squarefree and has no common divisor with  $f$ . We have then

$$\tau(\chi) = \mu\left(\frac{m}{f}\right)\psi\left(\frac{m}{f}\right)\tau(\psi) \tag{17}$$

and

$$\mathcal{G}(\alpha, \chi) = g(\alpha)\tau(\chi) \tag{18}$$

On the other hand, if  $m/f$  is not square free or has a common divisor with  $f$ , then the right hand side of (17) is zero. So, (17) holds for any character  $\chi$ . For another proof of this see [4], p. 148.

(b) If  $\chi = \chi_0 =$  principal character mod  $m$ , then  $f = 1$ ,  $\psi \equiv 1$ ,  $\tau(\psi) = 1$ ,  $q = \tilde{m} =$  squarefree kernel of  $m$ ,  $R = m/\tilde{m}$ , and  $\mathcal{G}(\alpha, \chi_0) = C_m(\alpha) =$  RAMANUJANS SUM.

Theorem *B* gives the well-known formula:

$$C_m(\alpha) = 0 \quad \text{if} \quad \frac{m}{\tilde{m}} \nmid \alpha$$

$$C_m\left(\frac{m}{\tilde{m}}n\right) = \frac{m}{\tilde{m}}\mu(\tilde{m})\mu((n, \tilde{m}))\varphi((n, \tilde{m})).$$

From (17) we get for all  $m$

$$C_m(1) = \mu(m).$$

5. Remarks: (a) It is clear that  $\mathcal{G}(\alpha, \chi)$  cannot vanish identically. So by 4. (a), formula (1) can only hold if  $R = 1$ , and if  $g(\alpha) = \bar{\chi}(\alpha)$  for all  $\alpha$ . But this is only possible if  $q = 1$ , i.e. if  $m = f$ . This shows that (1) characterises primitive characters, a fact proved by T.M. Apostol [5].