2. Proof of theorem B

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **23 (1977)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **12.07.2024**

THEOREM B. Let m, χ, f, ψ be as above. We put

$$q = \prod_{\substack{p \mid m \\ p \nmid f}} p \,, \quad R = \frac{m}{fq}$$

where p denotes rational primes. Let the multiplicative function g be defined by

$$g(n) = \mu((n, q))\, \varphi((n, q))\, \bar{\psi}(n)\,.$$

Then we have:

$$\left.\begin{aligned} \mathscr{G}(\alpha, \chi) &= 0 \text{ if } R \nmid \alpha \\ \mathscr{G}(Rn, \chi) &= \mu(q)\, \psi(q)\, \tau(\psi)\, R\, g(n), \quad n = 1, 2, \ldots \end{aligned}\right\} \qquad (3)$$

2. PROOF OF THEOREM B

For a Dirichlet character χ mod m let the function $L(s, \chi)$ be given by

$$L(s, \chi) = \sum_{1}^{\infty} \chi(n)\, n^{-s}\,, \quad Re\, s > 1\,.$$

The series defines an analytic function for Re $s > 1$, which can be extended to a meromorphic function on the whole complex plane, with at most one simple pole at $s = 1$. If χ is primitive, then $L(s, \chi)$ satisfies the equation

$$L(1-s, \chi) = m^{s-1} (2\pi)^{-s}\, \Gamma(s) \left(e^{-\frac{\pi i s}{2}} + \chi(-1)\, e^{\frac{\pi i s}{2}}\right) \tau(\chi)\, L(s, \bar{\chi})\,. \qquad (4)$$

Because of (1), this can also be written, for Re $s > 1$, as

$$L(1-s, \chi) = m^{s-1} (2\pi)^{-s}\, \Gamma(s) \left(e^{-\frac{\pi i s}{2}} + \chi(-1)\, e^{\frac{\pi i s}{2}}\right) \sum_{1}^{\infty} \mathscr{G}(n, \chi)\, n^{-s}\,. \qquad (5)$$

Whereas (4) holds only for primitive characters, (5) turns out to be valid in the general case. In fact, a much more general formula is proved in [3], th. 6.1; if we put there $x = \alpha = 0$, $\alpha_n = \chi(n)$ and observe that $\mathscr{G}(-n, \chi) = \chi(-1)\, \mathscr{G}(n, \chi)$, we get (5) immediately. But also most of the classical (= non-adelic) proofs of (4) will give (5) after very small changes. The only use of the primitivity of χ in these proofs is that they replace $\mathscr{G}(n, \chi)$ by $\bar{\chi}(n)\, \tau(\chi)$ at some stage (See for instance [7]).

Now let χ, m, ψ, f be as in the theorem. We have, by the Euler-product,

$$\begin{aligned} L(1-s, \chi) &= L(1-s, \psi) \prod_{p \mid q} \left(1 - \frac{\psi(p)}{p^{1-s}}\right) \\ &= L(1-s, \psi)\, \mu(q)\, q^{s-1}\, \psi(q) \prod_{p \mid q} (1 - p\bar{\psi}(p)\, p^{-s}) \end{aligned} \qquad (6)$$

Formula (4) is valid if we write f for m and ψ for χ. Eliminating $L(1-s, \chi)$ and $L(1-s, \psi)$ out of the equations (4), (5), (6), and taking into account that $\chi(-1) = \psi(-1)$, we obtain after an easy calculation, for Re $s > 1$:

$$\sum_{1}^{\infty} \mathscr{G}(n, \chi)\, n^{-s} = R\,\tau(\psi)\,\mu(q)\,\psi(q)\,R^{-s}\,L(s, \bar{\psi}) \prod_{p|q} \left(1 - \frac{p\bar{\psi}(p)}{p^{s}}\right) \tag{7}$$

Using Euler's product, we have for Re $s > 1$

$$\begin{aligned} L(s, \bar{\psi}) \prod_{p|q} (1 - p\bar{\psi}(p)\, p^{-s}) & \\ = \prod_{p \nmid q} \sum_{k=0}^{\infty} \bar{\psi}(p^k)\, p^{-ks} \cdot \prod_{p|q} \Big(1 + (1-p) \sum_{k=1}^{\infty} \bar{\psi}(p^k)\, p^{-ks}\Big) & \\ = \prod_{p} \sum_{k=0}^{\infty} \bar{\psi}(p^k)\, \mu((p^k, q))\, \varphi((p^k, q))\, p^{-ks} & \\ = \prod_{p} \sum_{k=0}^{\infty} g(p^k)\, p^{-ks} = \sum_{1}^{\infty} g(n)\, n^{-s}\,. & \end{aligned}$$

If we put that into (7) we get

$$\sum_{1}^{\infty} \mathscr{G}(n, \chi)\, n^{-s} = R\,\tau(\psi)\,\mu(q)\,\psi(q) \sum_{1}^{\infty} g(n)\,(Rn)^{-s}\,,$$

and we obtain (3) by comparing the coefficients.

3. In this section we show that one can prove theorem A using theorem B and conversely. Let $K(\alpha)$ resp. $H(\alpha)$ be the right hand side of (2) resp. (3). We want to prove $K(\alpha) = H(\alpha)$. First we show that $K(\alpha) \neq 0$ iff $H(\alpha) \neq 0$.

Now $H(\alpha) \neq 0$ iff $R | \alpha$ and $(\alpha R^{-1}, f) = 1$, and this is equivalent to

$$(\alpha f q, f m) = m \tag{8}$$

On the other hand $K(\alpha) \neq 0$ iff the four conditions hold

$$(\alpha, m) f \mid m \tag{9}$$

$$\frac{m}{(\alpha, m) f} \quad \text{is squarefree} \tag{10}$$

$$\left(\frac{m}{(\alpha, m) f}, f\right) = 1 \tag{11}$$

$$\left(\frac{\alpha}{(\alpha, m)}, f\right) = 1 \tag{12}$$

At several places in the following we will use that q is squarefree, $(f, q) = 1$ and the prime divisors of m are precisely the prime divisors of fq.

Let us assume (8). Then $f(\alpha, m) \mid f(\alpha q, m) = m$. Also $\frac{m}{f(\alpha, m)} = \frac{(\alpha q, m)}{(\alpha, m)} \Bigm| q$. This proves (9) and (10). From $f = m/(\alpha q, m)$ we have

$$\bigl(\alpha/(\alpha, m), f\bigr) = \bigl(\alpha/(\alpha, m), m/(\alpha q, m)\bigr) \mid \bigl(\alpha/(\alpha, m), m/(\alpha, m)\bigr) = 1 .$$

This proves (12). Finally

$$\left(\frac{m}{(\alpha, m) f}, f\right) = \left(\frac{f(\alpha q, m)}{(\alpha, m) f}, f\right) \Bigm| (q, f) = 1 ,$$

proving (11).

Conversely assume (9)-(12). From (10) and (11) we infer that $\frac{m}{f(\alpha, m)} \Bigm| q$. This implies

$$m \mid m \left(\frac{\alpha}{(\alpha, m)}, f\right) = \left(\frac{m\alpha f}{f(\alpha, m)}, mf\right) \Bigm| (q\alpha f, mf) . \tag{13}$$

Also

$$f(m, q\alpha) = f(\alpha, m)\left(\frac{m}{(\alpha, m)}, q\,\frac{\alpha}{(\alpha, m)}\right) = f(\alpha, m)\left(\frac{m}{(\alpha, m)}, q\right) .$$

In the last term, the numbers f and $(m/(\alpha, m), q)$ both divide $m/(\alpha, m)$, because of (9), and they are coprime, hence their product divides $m/(\alpha, m)$. This gives $f(m, q\alpha) \mid m$. Together with (13) this implies (8).

It remains to prove that $H(\alpha) = K(\alpha)$ for $\alpha = Rn$, $(n, f) = 1$. We have $m_0 = m/(\alpha, m) = f q/(n, fq) = fq/(n, q)$. Hence

$$R = \frac{m}{fq} = \frac{\varphi(m)}{\varphi(fq)} = \frac{\varphi(m)}{\varphi(f)\,\varphi(q)} = \frac{\varphi(m)}{\varphi(f)\,\varphi\bigl((n, q)\bigr)\,\varphi\left(\frac{q}{(n, q)}\right)}$$

$$= \frac{\varphi(m)}{\varphi\bigl((n, q)\bigr)\,\varphi(m_0)}$$

hence

$$R\,\varphi\bigl((n, q)\bigr) = \varphi(m)/\varphi(m_0) \tag{14}$$

Also $\mu(q) = \mu\bigl((n, q)\bigr)\,\mu\bigl(q/(n, q)\bigr)$, so

$$\mu(q)\,\mu\bigl((n, q)\bigr) = \mu\bigl(q/(n, q)\bigr) = \mu(m_0/f) . \tag{15}$$

Finally $\alpha_0 = n/(n, q)$, so $\alpha_0 q = n m_0/f$,
hence

$$\psi(\alpha_0)\psi(q) = \psi(m_0/f)\psi(n)$$

or

$$\bar{\psi}(n)\psi(q) = \bar{\psi}(\alpha_0)\psi\left(\frac{m_0}{f}\right) \tag{16}$$

Multiplying (14), (15) and (16) we find $K(\alpha) = H(\alpha)$.

4. Special cases

(a) Theorem B implies that $\tau(\chi) \neq 0$ if and only if $R = 1$, that is, if and only if m/f is squarefree and has no common divisor with f. We have then

$$\tau(\chi) = \mu\left(\frac{m}{f}\right)\psi\left(\frac{m}{f}\right)\tau(\psi) \tag{17}$$

and

$$\mathcal{G}(\alpha, \chi) = g(\alpha)\tau(\chi) \tag{18}$$

On the other hand, if m/f is not square free or has a common divisor with f, then the right hand side of (17) is zero. So, (17) holds for any character χ. For another proof of this see [4], p. 148.

(b) If $\chi = \chi_0 =$ principal character mod m, then $f = 1$, $\psi \equiv 1$, $\tau(\psi) = 1$, $q = \tilde{m} =$ squarefree kernel of m, $R = m/\tilde{m}$, and $\mathcal{G}(\alpha, \chi_0) = C_m(\alpha) =$ Ramanujans sum.

Theorem B gives the well-known formula:

$$C_m(\alpha) = 0 \quad \text{if} \quad \frac{m}{\tilde{m}} \nmid \alpha$$

$$C_m\left(\frac{m}{\tilde{m}}n\right) = \frac{m}{\tilde{m}}\mu(\tilde{m})\mu((n, \tilde{m}))\varphi((n, \tilde{m})).$$

From (17) we get for all m

$$C_m(1) = \mu(m).$$

5. Remarks: (a) It is clear that $\mathcal{G}(\alpha, \chi)$ cannot vanish identically. So by 4. (a), formula (1) can only hold if $R = 1$, and if $g(\alpha) = \bar{\chi}(\alpha)$ for all α. But this is only possible if $q = 1$, i.e. if $m = f$. This shows that (1) characterises primitive characters, a fact proved by T.M. Apostol [5].