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ON THE NUMBER OF ZEROS OF FUNCTIONS

by A. J. van der Poorten

0. Introduction

The object of this note is to give a complete description of a technique that leads to estimates for the number of zeros (always assumed to be counted according to multiplicity) of certain classes of functions in discs of given radius and centre in the complex plane. As we show, the technique also suffices to prove that functions cannot be (relatively) small too often in discs. In order that this paper may be a useful source we have made our proofs essentially self-contained; our lemmas are often more general than is required for the immediate applications and we have taken the opportunity to mention various formulae and tricks which, though no doubt well-known in the folklore, are by no means readily accessible in the literature.

We principally consider the case of exponential polynomials, that is, solutions of homogeneous linear differential equations with constant coefficients, and then briefly indicate the manner in which the method described extends to a very much wider class of functions.

Though the results are of general interest, the principal motive for their formulation has resided in their application in the theory of transcendental numbers. In this context one constructs auxiliary functions and shows that the contradiction of the result to be proved implies that, contrary to the construction, the auxiliary function vanishes identically; see, Gelfond [7], Chapter III, Tijdeman [28], Brownawell [2], Waldschmidt [34], Cudnovskii [3] (see Waldschmidt [35] for a summary) for typical application of theorem 1. The second result, theorem 2 is important in obtaining transcendence measures as well as in recent work on algebraic independence; for a recent application see, for example Cijsouw [4].

The present theory would seem to have been initiated by the work of Gelfond, see [7], p. 140ff. The work of Tijdeman [26], see also [27], pıovided the major breakthrough which has simplified subsequent results.

There is also an analogous p-adic theory, see for example Shorey [25]. In fact the results are simpler in the p-adic case as can be seen in the recent work of van der Poorten [24], see also [23].

1. A basic lemma

One learns that the essential step in constructing an estimate for the number of zeros of a function in a given disc consists of obtaining an upper bound for a ratio

$$|F|_{S^*} / |F|_S , \tag{1}$$

where $S^* > S > 0$, and, if the given disc has centre z_0, then $|F|_R = \max_{|z-z_0|=R} |F(z)|$. We see that this is sufficient by virtue of the following lemma, (see Waldschmidt [36], p. 166, for a slightly weaker statement).

Lemma 1. *Let* S^*, S, R *be real numbers satisfying*

$$S^* > S > 0 \quad and \quad S^* \geqslant R > 0 .$$

Let F *be a function holomorphic in some open set containing the disc* $|z - z_0| \leqslant S^*$. *If* F *does not vanish identically in the disc* $|z - z_0| \leqslant S^*$ *then the number of zeros* $n(F, R, z_0)$ *of* F *in the disc* $|z - z_0| \leqslant R$ *satisfies*

$$n(F, R, z_0) \operatorname{Log} \left(\frac{S^{*2} + SR}{S^*(S+R)} \right) \leqslant \operatorname{Log} \frac{|F|_{S^*}}{|F|_S} \tag{2}$$

Proof. There is no loss of generality in supposing for convenience that $z_0 = 0$. Suppose then that F has zeros at $z_1, \ldots, z_n$ in the disc $|z| \leqslant R$ and write

$$G(z) = F(z) \prod_{h=1}^{n} \frac{S^{*2} - z\bar{z}_h}{S^*(z - z_h)} .$$

Then G is holomorphic in an open set containing the disc $|z| \leqslant S^*$, a simple calculation confirms that

$$|G|_{S^*} = |F|_{S^*},$$

and, by the maximum-modulus principle,

$$|G|_S \leqslant |G|_{S^*}$$

However

$$|G|_S \geqslant |F|_S \prod_h \min \left| \frac{S^{*2} - Se^{i\theta} \bar{z}_h}{S^*(Se^{i\theta} - z_h)} \right|$$

Each factor in the product on the right is the square root of an expression of the shape

$$(3) \qquad (S^{*4} - 2SR_h S^{*2} \cos\psi + S^2 R_h^2)/S^{*2}(S^2 - 2SR_h \cos\psi + R_h^2),$$

where $z_h = R_h e^{i\phi_h}$ and $\psi = \theta - \phi_h$. One sees that the turning points of (3) as a function of ψ occur when $\sin\psi = 0$ and that the minimal value of (3) is

$$(4) \qquad ((S^{*2} + SR_h)/S^*(S+R_h))^2$$

One easily confirms that (4) is minimal for $0 \leqslant R_h \leqslant R$ when $R_h = R$, whence we obtain

$$|G|_S \geqslant |F|_S \left(\frac{S^{*2} + SR}{S^*(S+R)}\right)^n,$$

and the assertion of the lemma follows.

The lemma is " best possible "; the function $F(z) = \left(\frac{S^*(R-z)}{S^{*2} - Rz}\right)^n$ being the extreme case. I am indebted to Michel Waldschmidt for mentioning the result of the lemma to me. The lemma improves upon a similar result obtainable via Jensen's theorem, (see, for example, Tijdeman [26], p. 3).

According to the above observations, our principal attention below is directed towards the finding of upper bounds for ratios of the shape (1). Although the principles of our techniques are not new, many of the details have been little more than folklore and are presented here explicitly for the first time.

2. A USEFUL IDENTITY

The following lemma is presented in somewhat exaggerated generality. Its implications will become clear when below we come to look at specific examples.

LEMMA 2. *Let* S^*, S *be real numbers satisfying* $S^* \geqslant S > 0$ *and let* G *be a function of the shape*

$$G(z) = \textstyle\sum_{k=1}^{\sigma} b_k g_k(z),$$

$b_1, \ldots, b_\sigma$ *complex constants, where* $g_1, \ldots g_\sigma$ *are functions holomorphic in some open set containing the disc* $|z - z_0| \leqslant S^*$. *Further let* $z_1, \ldots, z_\sigma$ *be points in the disc* $|z - z_0| < S$ *and let* $t_1, \ldots t_\sigma$ *be non-negative integers.*

Finally denote by Δ_{ji} *the cofactor of the typical element in the* $\sigma \times \sigma$ *determinant*

$$\Delta = | g_i^{(t_j)}(z_j) |_{1 \leq i,j \leq \sigma},$$

suppose that $\Delta \neq 0$, *and assume the notational conventions of the introduction above. Then for* w *such that* $|w - z_0| = S^*$ *we have*

$$(5) \quad G(w) = \sum_{\lambda=1}^{\sigma} \sum_{k=1}^{\sigma} \left\{ \frac{1}{2\pi i} \int_{|\gamma - z_0| = S} \frac{\Delta_{\lambda,k}}{\Delta} g_k(w) \left(\frac{d}{dz} \right)^{t_\lambda} G(\gamma) \frac{d\gamma}{(\gamma - z)} \right\}_{z = z_\lambda}$$

and it follows that if G *does not vanish identically*

$$(6) \quad |G|_{S^*} / |G|_S \leqslant \sum_{\lambda=1}^{\sigma} \max \left| \sum_{k=1}^{\sigma} \frac{\Delta_{\lambda,k}}{\Delta} g_k(w) \right| \cdot S \frac{t_\lambda!}{(S - |z_\lambda - z_0|)^{t_\lambda + 1}}$$

Proof. By the residue theorem the right-hand side of (5) is

$$\sum_{\lambda=1}^{\sigma} \sum_{k=1}^{\sigma} \frac{\Delta_{\lambda,k}}{\Delta} g_k(w) G^{(t_\lambda)}(z_\lambda)$$

$$= \sum_{k=1}^{\sigma} \sum_{h=1}^{\sigma} b_h g_k(w) \sum_{\lambda=1}^{\sigma} g_h^{(t_\lambda)}(z_\lambda) \frac{\Delta_{\lambda,k}}{\Delta}$$

$$= \sum_{k=1}^{\sigma} \sum_{h=1}^{\sigma} b_h g_k(w) \delta_{hk}, \quad (\delta_{hk}, \quad \text{the Kronecker delta})$$

$$= G(w),$$

as was asserted. Having thus established the identity (5), we conclude that

$$|G|_{S^*} \leqslant \sum_{\lambda=1}^{\sigma} \max \left| \sum_{k=1}^{\sigma} \frac{\Delta_{\lambda,k}}{\Delta} g_k(w) \left| \frac{1}{2\pi} \oint G(\gamma) \frac{t_\lambda!}{(\gamma - z_\lambda)^{t_\lambda + 1}} \right| \right|$$

and estimating the integral on the circle $|\gamma - z_0| = S$, the assertion (6) is immediate.

We have stated the lemma in such generality as might be appropriate for the purposes of this note. The reader should observe that, moreover, the same idea can be used to obtain any combination

$$\sum_{k=1}^{\sigma} b_k h_k$$

on the left-hand side of an identity similar to (5); this is useful in isolating the coefficients b_k which is necessary when one is investigating the number of points in a disc at which the given function $G(z)$ may be small; see theorem 2 below for details. We remark that the identity (5) should be viewed as a (degenerate) case of the integral form of the Hermite interpolation formula.

3. An estimation by interpolation

Lemma 2 reduces the problem of estimating the number of zeros to one of finding an upper bound for determinantal combinations of the shape

$$\left| \sum_{k=1}^{\sigma} \frac{\Delta_{\lambda,k}}{\Delta} g_k(w) \right| .$$

As we propose to discuss only some very special cases, we alert the reader on the one hand to the encyclopaedic Muir [14], and, for some determinants relevant in transcendence work, to van der Poorten [21].

Lemma 3. *Let* $\omega_1, \ldots, \omega_\sigma$ *be complex numbers and denote by* $D_{j,i}$ *the cofactor of the typical element in the* $\sigma \times \sigma$ *determinant*

$$D = | \omega_i^{j-1} |_{1 \leq i,j \leq \sigma} .$$

Let n *be a positive integer, and write* $\max_k | \omega_k | \leqslant \Omega$. *Then for each* $\lambda = 1, 2, \ldots, \sigma$

$$\left| \sum_{k=1}^{\sigma} \frac{D_{\lambda,k}}{D} \frac{(\omega_k w)^{n-1}}{(n-1)!} \right| \leqslant \frac{1}{\Omega^{\lambda-1}} \sum_{h=1}^{\sigma} \frac{(\Omega |w|)^{h-1}}{(h-1)!} \frac{(\Omega |w|)^{n-h}}{(n-h)!} \binom{h-1}{\lambda-1} \tag{7}$$

Note. The quantity on the left of (7) remains well-defined by continuity even though the ω_k be not distinct. However, we treat the ω_k as formally distinct.

Proof. We commence by asserting that $\sum_{k=1}^{\sigma} \frac{D_{\lambda,k}}{D} \omega_k^{n-1}$ is the coefficient of $z^{\lambda-1}$ in the polynomial

$$P(z) = \sum_{k=1}^{\sigma} \omega_k^{n-1} \prod_{\substack{h=1 \\ h \neq k}}^{\sigma} \left(\frac{z - \omega_h}{\omega_k - \omega_h} \right) \tag{8}$$

To see this, observe that $P(z)$ is the unique polynomial of degree at most $\sigma - 1$ determined by the σ conditions (this is just Lagrange interpolation)

$$P(\omega_h) = \omega_h^{n-1}, \; (h = 1, \ldots, \sigma) . \tag{9}$$

On the other hand, if

$$Q(z) = \sum_{\lambda=1}^{\sigma} \left(\sum_{k=1}^{\sigma} \frac{D_{\lambda,k}}{D} \omega_k^{n-1} \right) z^{\lambda-1} ,$$

then

$$Q(\omega_h) = \sum_{k=1}^{6} \omega_k^{n-1} \left(\sum_{\lambda=1}^{\sigma} \omega_h^{\lambda-1} \frac{D_{\lambda,k}}{D} \right) = \sum_{k=1}^{\sigma} \omega_k^{n-1} \delta_{kh} = \omega_h^{n-1},$$

and it follows that $Q(z) \equiv P(z)$ as asserted.

To now evaluate the coefficients of $P(z)$ we expand P in a Newton interpolation series

(10) $$P(z) = \sum_{h=1}^{\sigma} b_h (z - \omega_1) \dots (z - \omega_{h-1}),$$

and observe that by virtue of the residue formula we actually have

$$b_h = \frac{1}{2\pi i} \int_C \frac{P(\gamma)}{(\gamma - \omega_1) \dots (\gamma - \omega_h)} d\gamma = \frac{1}{2\pi i} \int_C \frac{\gamma^{n-1}}{(\gamma - \omega_1) \dots (\gamma - \omega_h)} d\gamma,$$
$$(h = 1, \dots, \sigma),$$

where the contour C is, say, any circle about the origin of sufficiently large radius in order that C contain the points $\omega_1, \dots, \omega_\sigma$. The second, rather remarkable, equality is of course a consequence of the fact that the residue formula only " notices " P at the poles $\omega_1, \dots, \omega_h$, and at these points, (8) implies (9), so $P(\gamma)$ coincides with γ^{n-1}.

It is convenient to evaluate the second integral at its pole (if there is indeed such a pole) at ∞. Accordingly we obtain

(11) $$b_h = \frac{1}{2\pi i} \int_C \frac{\gamma^{n-1}}{(\gamma - \omega_1) \dots (\gamma - \delta_h)} d\gamma = \frac{1}{2\pi i} \int_{C'} \frac{d\gamma}{\gamma^{n-h+1} (1 - \omega_1 \gamma) \dots (1 - \omega_h \gamma)}$$

where C' is now a circle about the origin of sufficiently small radius in order that C' not contain the points $\omega_1^{-1}, \dots, \omega_\sigma^{-1}$ (if some ω_k should vanish treat it as formally nonzero albeit arbitrarily small). It follows that b_h is exactly the coefficient of γ^{n-h} in the power series expansion about the origin of $\{ (1 - \omega_1 \gamma) \dots (1 - \omega_h \gamma) \}^{-1}$, that is

(12) $$| b_h | = | \sum_{|\mu| = n-h} \omega_1^{\mu(1)} \dots \omega_h^{\mu(h)} | \leqslant \binom{n-1}{h-1} \Omega^{n-h}.$$

It is now no longer of any matter that the ω_k not be distinct or that any should vanish. Inserting the estimate (12) in (10) we easily see that

(13) $$\sum_{h=1}^{\sigma} \binom{h-1}{\lambda-1} \Omega^{h-\lambda} \binom{n-1}{h-1} \Omega^{n-h}$$

is an upper bound for the coefficient of $z^{\lambda-1}$ in the polynomial $P(z)$ of (8). Accordingly we have that

$$\left| \sum_{k=1}^{\sigma} \frac{D_{\lambda,k}}{D} \frac{(\omega_k w)^{h-1}}{(n-1)!} \right| \leqslant \sum_{h=1}^{\sigma} \frac{|w|^{n-1}}{(n-1)!} \binom{n-1}{h-1} \binom{h-1}{\lambda-1} \Omega^{n-\lambda}$$

$$= \frac{1}{(\lambda-1)!} \sum_{h=1}^{\sigma} \frac{\Omega^{h-\lambda}}{(h-\lambda)!} \frac{(\Omega |w|)^{n-h}}{(n-h)!} |w|^{h-1},$$

which is the assertion.

The following is essentially an immediate corollary of the previous lemma.

LEMMA 4. *Let* g *be a function analytic in a sufficiently large disc about the origin and suppose that in that disc*

$$g(z) = \sum_{n=1}^{\infty} \frac{c_{n-1}}{(n-1)!} z^{n-1} . \tag{14}$$

Let $g_k(z) = g(\omega_k z)$, $(k=1, \ldots, \sigma)$ and *otherwise let the notation be as in lemma 3. Then if* $|g|$ *is the function*

$$|g|(z) = \sum_{n=1}^{\infty} \frac{|c_{n-1}|}{(n-1)!} z^{n-1} , \tag{15}$$

we have for each $\lambda = 1, \ldots, \sigma$

$$\left| \sum_{k=1}^{\sigma} \frac{D_{\lambda,k}}{D} g(\omega_k w) \right| \leqslant \frac{1}{\Omega^{\lambda-1}} \sum_{h=1}^{\sigma} \frac{(\Omega |w|)^{h-1}}{(h-1)!} |g|^{(h-1)}(\Omega |w|) \binom{h-1}{\lambda-1} \tag{16}$$

Proof. By lemma 3 we have

$$\left| \sum_{k=1}^{\sigma} \frac{D_{\lambda,k}}{D} g(\omega_k w) \right| \leqslant \left| \sum_{k=1}^{\sigma} \sum_{n=1}^{\infty} \frac{D_{\lambda,k}}{D} c_{n-1} \frac{(\omega_k w)^{n-1}}{(n-1)!} \right|$$

$$\leqslant \frac{1}{\Omega^{\lambda-1}} \sum_{h=1}^{\sigma} \frac{(\Omega |w|)^{h-1}}{(h-1)!} \binom{h-1}{\lambda-1} \sum_{n=1}^{\infty} |c_{n-1}| \frac{(\Omega |w|)^{n-h}}{(n-h)!},$$

which is the assertion.

The critical aspect of the above estimates is that they are independent of $\min_{h \neq k} |\omega_k - \omega_h| = d$. The interpolation method of lemma 3 is not at all new nor is the idea of obtaining results independent of d. The latter seems appropriately attributable to Turán [30], whilst the former occurs in Makai [11], [12] in the context of our problem. The interpolation method appears in a more general way in the thesis of van der Poorten [16], and thence in the papers [17], [18] [19]. However the recognition of the general pattern is due to Tijdeman [26], whence see Balkema and Tijdeman [1]. For further details see the references cited in the papers mentioned above.

4. Exponential polynomials

We commence by making explicit some folklore the principles of which can be found in [16] and Tijdeman [26], and which is made explicit in another context in van der Poorten [20].

Lemma 5. *For some fixed positive integer* σ, *and some given function* g, *supposed holomorphic in the domain under consideration, denote by* J *the set of functions* G *of the shape*

$$G(z) = \sum_{k=1}^{\sigma} b_k\, g(\alpha_k z),$$

where $b_1, \ldots, b_\sigma; \alpha_1, \ldots, \alpha_\sigma$ *are complex numbers. Then, for all sets of non-negative integers* $\rho(1), \ldots, \rho(m)$ with *sum* $\sum_{h=1}^{m} \rho(h) = \sigma$ *(and all positive integers* m *such that* $1 \leqslant m \leqslant \sigma$*), for each function* F *of the shape*

$$F(z) = \sum_{h=1}^{m} \sum_{t=1}^{\rho(h)} a_{ht}\, z^{t-1} g^{(t-1)}(\omega_h z),$$

the a_{ht} *complex constants, there is a sequence of functions in* J *converging uniformly to* F *in compact sets.*

Proof. The lemma depends upon noticing that functions of the shape F are actually, in a sense, particular cases of, rather than generalisations of functions of the shape G. Indeed, reindex so that G appears as

$$G(z) = \sum_{h=1}^{m} \sum_{t=1}^{\rho(h)} b_{ht}\, g(\omega_{ht} z), \tag{17}$$

and choose the coefficients b_{ht} as functions of $\omega_{11}, \ldots, \omega_{m\rho(m)}$ (so of $\alpha_1, \ldots, \alpha_\sigma$) so that for each $h = 1, \ldots, m$

$$\sum_{t=1}^{\rho(h)} b_{ht}\, g(\omega_{ht} z) = \sum_{t=1}^{\rho(h)} a_{ht} \frac{(t-1)!}{2\pi i} \int_C g(\gamma z) \prod_{s=1}^{t} (\gamma - \omega_{hs})^{-1}\, d\gamma, \tag{18}$$

where the closed contour C contains all the ω_{ht} but excludes any singularities of g. Clearly there exists a sequence of σ-tuples $(\omega_{11}, \ldots, \omega_{m\rho(m)})$ which converges to $(\omega_1, \ldots, \omega_1; \omega_2, \ldots, \omega_m)$ componentwise, and in the limit, (18) shows that (17) becomes $F(z)$.

I am indebted to D. W. Masser for any felicities in the terminology used in the lemma.

The following theorem is a result due to Tijdeman [26], [27]; see also Waldschmidt [36] p. 164-174. A history of the problem can be found in the above mentioned works.

Definition. Let $\rho(1), \ldots, \rho(m)$ be non-negative integers with sum $\sum \rho(h) = \sigma$, let a_{ht} $(h=1, \ldots, m;\ t=1, \ldots, \rho(h))$ be complex numbers which do not all vanish, and let $\omega_1, \ldots, \omega_m$ be distinct complex numbers. Then a function of the shape

$$F(z) = \sum_{h=1}^{m} \sum_{t=1}^{\rho(h)} a_{ht}\, z^{t-1} e^{\omega_h z} \tag{19}$$

is called an *exponential polynomial* of *degree* σ, with *frequencies* ω_h and *coefficients* a_{ht}.

Theorem 1. *The number of zeros* $n(F, R, z_0)$ *of an exponential polynomial* F *of degree* σ *and with frequencies* $\omega_1, \ldots, \omega_m$ *satisfying* $\max_h |\omega_h| \leqslant \Omega$, *in a disc of centre* z_0 *and radius* R, *is less than*

$$\frac{\log \gamma}{\log \tau}(\sigma - 1) + \frac{(\gamma\tau - 1)(\gamma + 1)}{\gamma(\gamma - \tau)\log \tau}\,\Omega R + \frac{1}{\log \tau\,(\gamma - 1)}, \tag{20}$$

for all $\gamma > \tau > 1$.

Proof. We consider the exponential sum

$$G(z) = \sum_{k=1}^{\sigma} b_k e^{\omega_k z}, \quad \max_k |\omega_k| \leqslant \Omega,$$

and suppose that $z_0 = 0$. In lemma 2 take $g_k(z) = e^{\omega_k z}$ and t_j, z_j such that $t_\lambda = \lambda - 1$, $z_\lambda = 0$ $(\lambda = 1, \ldots, \sigma)$ and observe that the determinant Δ of lemma 2 now coincides with the Vandermonde determinant D of lemmas 3 and 4. Then from lemmas 2 and 4 we obtain

$$|G|_{S^*}/|G|_S \leqslant \sum_{\lambda=1}^{\sigma} \left(\frac{S^*}{S}\right)^{\lambda-1} \sum_{h=1}^{\sigma} \frac{(\Omega S^*)^{h-\lambda}}{(h-\lambda)!}\, e^{\Omega S^*}. \tag{21}$$

We observe that the information (21) is independent of the coefficients of G and independent of $\min_{h \neq k} |\omega_k - \omega_h|$; furthermore, under a translation only the coefficients of G change, so (21) is valid for all centres z_0. So by lemma 5 we have, writing $S^*/S = \gamma > 1$,

$$\begin{aligned} |F|_{S^*}/|F|_S &\leqslant e^{\Omega S^*} \sum_{h=0}^{\sigma-1} \frac{(\Omega S^*)^h}{h!} \sum_{\lambda=1}^{\sigma-h} \gamma^{\lambda-1} \\ &= \frac{e^{\Omega S^*}}{\gamma - 1}\left(\gamma^\sigma \sum_{h=0}^{\sigma-1} \frac{(\Omega S)^h}{h!} - \sum_{h=0}^{\sigma-1} \frac{(\Omega S^*)^h}{h!}\right) \end{aligned} \tag{22}$$

It follows that, with the gain of some tidiness, but the loss of some precision, $\left(\text{and noticing that} \quad -\log\left(1-\frac{1}{\gamma}\right) < \frac{1}{\gamma-1}\right)$,

$$\log |F|_{S^*}/|F|_S < (\sigma-1)\log\gamma + \frac{1}{\gamma-1} + \Omega(S^*+S)\,.$$

Finally let $\tau = (S^{*2}+SR)/S^*(S+R) > 1$; then lemma 1 implies that

$$n(F, R, z_0) < \frac{1}{\log\tau}\left\{(\sigma-1)\log\gamma + \frac{1}{\gamma-1} + \Omega R\,\frac{(\gamma\tau-1)(\gamma+1)}{\gamma(\gamma-\tau)}\right\} \tag{23}$$

as asserted.

COROLLARY 1. *If* $\sigma = 1$ *then* $n(F, R, z_0) = 0$. *For* $\sigma > 1$ *we have*

$$n(F, R, z_0) < 3(\sigma-1) + 4\Omega R\,, \qquad \text{(Tijdeman [27]} \tag{24}$$

or

$$n(F, R, z_0) < 2(\sigma-1) + 5\Omega R\,, \qquad \text{(Waldschmidt [36])} \tag{25}$$

or

$$n(F, R, z_0) < 4(\sigma-1) + 3\Omega R\,. \tag{26}$$

Proof. The first remark is trivial; we require $\sigma > 1$ in order to assimilate the term $1/(\gamma-1)$ in (20). To obtain (24) choose, say, $\tau = 3.5$, $\gamma = 30$, and for (25) $\tau = 3.5$, $\gamma = 10$, whilst for (26) $\tau = 3.5$, $\gamma = 110$. Parameters were calculated on the *HP* 65 belonging to John Conway, for whom see Knuth [9].

Notwithstanding the apparent precision of our method, (20) gives quite inadequate results in the asymptotic cases. For example, we know from results of Pólya [15] and Dickson [6] that $\lim_{R\to\infty} n(F, R)/R < \Omega$, but (22) does no better than $\lim_{R\to\infty} n(F, R)/R \leqslant e\Omega$. At the opposite extreme, " the local valency problem ", M. Voorhoeve has shown, using an idea of Hayman [8], that if $\sigma \geqslant 4$ then $R \leqslant 1/8\,\Omega$ implies $n(F, R) \leqslant \sigma - 1$, but nothing like this precision is available from (20); incidentally, because F has σ coefficients, it is clear that in every disc, no matter how small, one may have $n(F, R) \geqslant \sigma - 1$.

Although theorem 1 is more than adequate for applications to transcendence arguments, one can do better; for example Voorhoeve [31] has shown that $n(F, R, z_0) \leqslant 2(\sigma-1) + \frac{4}{\pi}\Omega R$ by a quite different argument. Actually because the result in the exponential polynomial case is independent of centre z_0, lemma 1 is quite crude (note the " extreme case ") because it assumes that the zeros accumulate at a point near the edge of the disc.

We now turn to a generalisation of theorem 1 wherein we show that an exponential polynomial cannot be small at too many points. This result largely includes earlier similar results of Mahler [10], Tijdeman [29] and Cijsouw and Tijdeman [5]. As we shall see, theorem 2 actually contains theorem 1 as a special case.

We shall retain, without further explanation, the notations introduced in the lemmas above. In preparation for the proof of the theorem we require two lemmas.

LEMMA 6. *Let*

$$F(z) = \sum_{k=1}^{m} \sum_{s=1}^{\rho(k)} \frac{b_{ks}}{(s-1)!} z^{s-1} e^{\omega_k z}$$

be an exponential polynomial of degree at most $\sigma = \sum_{k=1}^{m} \rho(k)$.

Write

$$D_k = \Big| \prod_{\substack{h=1 \\ h \neq k}}^{m} (\omega_k - \omega_h)^{\rho(h)} \Big| , \quad \Omega_k = | \omega_k | , \quad d_k = \min_{h \neq k} | \omega_k - \omega_h |$$

$(k = 1, \ldots, m)$

Denote by θ *a real number such that* $(S\theta)^{\sigma-1} \geqslant (\sigma-1)\,!$ *and by* S *a real number we shall suitably determine below. Then*

$$| b_{ks} | \leqslant 2^{\sigma - s} D_k^{-1} \prod_{h=1}^{m} (\theta + \Omega h)^{\rho(h) - \delta_{hk}} d_k^{-(\rho(k)-s)} | F |_S$$

and in particular

$$| b_{k\rho(k)} | \leqslant D_k^{-1} \prod_{h=1}^{m} (\theta + \Omega h)^{\rho(h) - \delta_{hk}} | F |_S \quad (k = 1, \ldots m; s = 1, \ldots, \rho(k))$$

Proof. Notice (compare lemma 2) that if $\Delta_{\lambda,\, ks}$ is the cofactor of the typical element in the $\sigma \times \sigma$ determinant

$$\Delta = \left| \binom{\lambda - 1}{s - 1} \omega_k^{\lambda - s} \right|_{ks, \lambda}$$

(here rows are indexed by the pairs (k, s) and columns by λ) then plainly

$$b_{ks} = \sum_{\lambda=1}^{\sigma} \left\{ \frac{(\lambda-1)!}{2\pi i} \int_{|\zeta| = S} \frac{\Delta_{\lambda,ks}}{\Delta} F(\zeta) \frac{d\zeta}{\zeta^\lambda} \right\},$$

so

$$| b_{ks} | \leqslant | F |_S \sum_{\lambda=1}^{\sigma} \left| \frac{\Delta_{\lambda,ks}}{\Delta} \right| \frac{(\lambda-1)!}{S^{\lambda-1}} . \tag{27}$$

But

$$\sum_{\lambda=1}^{\sigma} \frac{(\lambda-1)!}{(\lambda-t)!}\, \omega_h^{\lambda-t} \frac{\Delta_{\lambda,ks}}{\Delta} = \begin{cases} (s-1)! & (h,t) = (k,s) \\ 0 & (h,t) \neq (k,s) \end{cases}$$

so $\Delta_{\lambda,ks}/\Delta$ is exactly the coefficient of $z^{\lambda-1}$ in the polynomial $P_{ks}(z)$ of degree at most $\sigma - 1$ defined by the σ conditions

$$P_{ks}^{(t-1)}(\omega_h) = \begin{cases} (s-1)! & (h,t) = (k,s) \\ 0 & (h,t) \neq (k,s) \end{cases}$$

We now make a change of scale whereby we replace z in $F(z)$ by $z\,\theta$; this is tantamount to replacing each ω_k by ω_k/θ whilst each b_{ks} become b_{ks}/θ^{s-1}. We arrange that $(S\theta)^{\sigma-1} \geqslant (\sigma-1)!$. Then (27) implies that it suffices to find an upper bound for the sum of absolute values of the coefficients of $P_{ks}(z)$. There is a useful stratagem whereby one obtains such a bound, dependent on the observation that if a polynomial $P(z)$ has non-negative real zeros then $|P(-1)|$ is the sum of the absolute values of its coefficients. For formal details of the required generalisation of this remark see van der Poorten [17], lemma 2.

One confirms readily that the polynomial $P_{ks}(z)$ is given by the integral

$$(28) \qquad P_{ks}(z) = -\frac{1}{2\pi i}\int_{C_k} \frac{(\zeta-\omega_k)^{s-1}}{\zeta - z} \prod_{h=1}^{m} \left(\frac{z-\omega_h}{\zeta-\omega_h}\right)^{\rho(h)} d\zeta$$

where C_k is a suitable contour about ω_k excluding the other ω_h and formally excluding z; in fact (28) is just a special case of an integral form of the Hermite interpolation formula.

So

$$P_{ks}(z) = D_k^{-1} \frac{(z-\omega_k)^{\rho(k)-1}}{(\rho(k)-s)!} \left(\frac{\partial}{\partial\zeta}\right)^{\rho(k)-s} \left\{ \prod_{\substack{h=1 \\ h\neq k}}^{m} \left\{ \frac{\dfrac{z-\omega_k}{\zeta-\omega_k}}{\dfrac{\zeta-\omega_k}{\omega_h-\omega_k} - 1} \right\}^{\rho(h)} \cdot \frac{1}{\zeta - z} \right\}_{\zeta=\omega_k}$$

whence

$$(29) \quad |P_{ks}(-1)| \leqslant D_k^{-1}\, 2^{\sigma-s} \prod_{h=1}^{m} (1+\Omega_h)^{\rho(h)-\delta_{hk}}\, d_k^{-(\rho(k)-s)}$$

But in the estimate (29) we have estimated the numerator as if $P_{ks}(z)$ were a sum of polynomials with non-negative real zeros $\Omega_1, \ldots, \Omega_m$. Hence (29) gives an upper bound for the sum of the absolute values of the coefficients of $P_{ks}(z)$. Recalling the scaling we are assuming, (27) implies that

$$|b_{ks}| \leqslant 2^{\sigma-s} D_k^{-1} \prod_{h=1}^{m} (\theta+\Omega_h)^{\rho(h)-\delta_{hk}}\, d_k^{-(\rho(k)-s)}\, |F|_S$$

In the special case $s = \rho(k)$ we have very simply that

$$P_{k\rho(k)}(z) = \prod_{h=1}^{m} \left(\frac{z-\omega_h}{\omega_k-\omega_h}\right)^{\rho(h)-\delta_{hk}}$$

and the estimate for $b_{k\rho(k)}$ follows immediately from (27) and the argument outlined above.

LEMMA 7. *Let* $\alpha_1, \ldots, \alpha_n$ *be distinct points such that*

$$\left|\frac{F^{(t-1)}(\alpha_h)}{(t-1)!}\right| \leqslant \chi_{ht} \leqslant \chi_h \leqslant \chi$$

and

$$\Delta_h = \Big|\prod_{\substack{k=1\\k\neq h}}^{m} (\alpha_h-\alpha_k)^{\tau(k)}\Big|, \quad |\alpha_h| = R_h \leqslant R, \quad \min_{k\neq h} |\alpha_h - \alpha_k| = \delta_h$$

$(h=1, \ldots, n;\ t=1, \ldots, \tau(h))$ *where* $\tau(1), \ldots, \tau(n)$ *are positive integers with sum* $\sum_{h=1}^{n} \tau(h) = N$.

Let S^*, S *be real numbers satisfying* $S^* > S > 0$ *and* $S^* \geqslant R > 0$.

Then.

$$|F|_S \leqslant \frac{S^*}{S^*-S} \prod_{k=1}^{n} \left(\frac{S^*(S+R_k)}{S^{*2}+SR_k}\right)^{\tau(k)} |F|_{S^*} +$$

$$+ \sum_{h=1}^{n} \sum_{t=1}^{\tau(h)} \chi_{ht} N^{-1} 3^{N-1} \Delta_h^{-1} \prod_{k=1}^{n} \left(\frac{(S^*+R_hR_k)(S+R_k)}{S^{*2}+SR_k}\right)^{\tau(k)} \delta_h^{-(\tau(h)-t)}$$

Proof. By an integral form of the Hermite interpolation formula we have

$$\frac{1}{2\pi i}\int_{|\zeta|=S} F(\zeta) \prod_{k=1}^{n} \left(\frac{S^{*2}-\zeta\bar{\alpha}_k}{S^*(\zeta-\alpha_k)}\right)^{\tau(k)} \frac{d\zeta}{\zeta - z}$$

$$= F(z) \prod_{k=1}^{n} \left(\frac{S^{*2}-z\bar{\alpha}_k}{S^*(z-\alpha_k)}\right)^{\tau(k)} +$$

$$+ \frac{1}{2\pi i} \sum_{h=1}^{n} \sum_{t=1}^{\tau(h)} \frac{F^{(t-1)}(\alpha_h)}{(t-1)!} \int_{C_h} (\zeta-\alpha_h)^{t-1} \prod_{k=1}^{n} \left(\frac{(S^{*2}-\zeta\bar{\alpha}_k}{(S^*(\zeta-\alpha_k)}\right)^{\tau(k)} \frac{d\zeta}{\zeta - z}$$

as can be seen directly by the Cauchy residue theorem; here C_h is a suitable contour about α_h excluding the other α_k and formally excluding z. By the argument detailed in lemma 2 we have for $|z| = S$.

$$(30) \qquad \left|\prod_{k=1}^{n} \left(\frac{S^{*2}-z\bar{\alpha}_k}{S^*(z-\alpha_k)}\right)^{\tau(k)}\right| \geqslant \prod_{k=1}^{n} \left(\frac{(S^{*2}+SR_k}{(S^*(S+R_k)}\right)^{\tau(k)}$$

and

$$(31) \qquad \left| \frac{1}{2\pi i} \int_{|\zeta| = S^*} F(\zeta) \prod_{k=1}^{n} \left(\frac{S^{*2} - \zeta \bar{\alpha}_k}{S^* (\zeta - \alpha_k)} \right)^{\tau(k)} \frac{d\zeta}{\zeta - z} \right| \leqslant |F|_{S^*} \frac{S^*}{S^* - S}$$

We select z such that $|F(z)| = |F|_s$ whence we may suppose that z is not near any α_h. Then by explicit evaluation similar to that in lemma 6 we obtain.

$$(32) \qquad \left| \frac{1}{2\pi i} \int_{C_h} (\zeta - \alpha_h)^{t-1} \prod_{k=1}^{n} \left(\frac{S^{*2} - \zeta \bar{\alpha}_k}{S^* (\zeta - \alpha_k)} \right)^{\tau(k)} \frac{d\zeta}{\zeta - z} \right|$$

$$\leqslant N^{-1} 3^{N-1} \Delta_h^{-1} \prod_{k=1}^{n} \left(\frac{S^{*2} + R_h R_k}{S^*} \right)^{\tau(k)} \delta_h^{-(\tau(h)-t)} .$$

The three inequalities (30), (31) and (32) together with the integral interpolation formula now readily yield the lemma.

THEOREM 2. *Let*

$$F(z) = \sum_{k=1}^{m} \sum_{s=1}^{\rho(k)} \frac{b_{ks}}{(s-1)!} z^{s-1} e^{\omega_k z}$$

be an exponential polynomial of degree at most $\sigma = \sum_{k=1}^{m} \rho(k) \quad (>1)$ *with*

$$D_k = \Big| \prod_{\substack{h=1 \\ h \neq k}}^{m} (\omega_k - \omega_h)^{\rho(h)} \Big| , \quad |\omega_k| = \Omega_k \leqslant \Omega ,$$

$$d_k = \min_{h \neq k} |\omega_k - \omega_h| \quad (k = 1, \dots, m)$$

Further let $\alpha_1, \dots, \alpha_n$ *be distinct points such that*

$$\Delta_h = \Big| \prod_{\substack{k=1 \\ k \neq h}}^{n} (\alpha_h - \alpha_k)^{\tau(k)} \Big| , \quad |\alpha_h| = R_h \leqslant R , \ \min_{k \neq h} |\alpha_h - \alpha_k| = \delta_h .$$

and

$$\left| \frac{F^{(t-1)}(\alpha_h)}{(t-1)!} \right| \leqslant \chi_{ht} \leqslant \chi_h \leqslant \chi$$

$(h=1, \dots n;\ t=1, \dots, \tau(h))$ *where* $\tau(1), \dots, \tau(n)$ *are positive integers with sum* $\sum_{h=1}^{n} \tau(h) = N.$

Then if

$$N \geqslant 2(\sigma - 1) + 5\Omega R$$

we have

$$| b_{ks} | < D_k^{-1} d_k^{-(\rho(k)-s)} (3N/4R)^{\sigma-1} \sum_{h=1}^{n} \sum_{t=1}^{\tau(h)} \chi_{ht} N^{-1} \Delta_h^{-1} \delta_h^{-(\tau(h)-t)} (5R)^N .$$

$$\leqslant D_k^{-1} d_k^{-(\rho(k)-s)} (3N/4R)^{\sigma-1} \Delta^{-1} (5R)^N \chi ,$$

(*where* $\Delta = \min_{(h,t)} \Delta_h \delta_h^{\tau(h)-t}$). $\qquad (k=1, \ldots, m;\ s=1, \ldots, \rho(k))$

In particular for $k = 1, \ldots, m$

$$| b_{k\rho(k)} | < D_k^{-1} (N/eR)^{\sigma-1} \sum_{h=1}^{n} \sum_{t=1}^{\tau(h)} \chi_{ht} N^{-1} \Delta_h^{-1} \delta_h^{-(\tau(h)-t)} (5R)^N$$

Proof. In the proof of theorem 1 we showed at (22) that

$$(33) \qquad | F |_{S^*} \leqslant | F |_S \frac{e^{\Omega S^*}}{\gamma - 1} \left(\left(\gamma^\sigma \sum_{h=0}^{\sigma-1} \frac{(\Omega S)^h}{h!} - \sum_{h=0}^{\sigma-1} \frac{(\Omega S^*)^h}{h!} \right) \right.$$

whilst lemma 7 gives an inequality

$$(34) \qquad | F |_S \leqslant | F |_{S^*} \frac{\gamma}{\gamma - 1} \prod_{k=1}^{n} \left(\frac{S^* (S + R_k)}{S^{*2} + SR_k} \right)^{\tau(k)} + E$$

with

$$E \leqslant \sum_{h=1}^{n} \sum_{t=1}^{\tau(h)} \chi_{ht} N^{-1} 3^{N-1} \Delta_h^{-1} \prod_{k=1}^{n} \left(\frac{(S^{*2} + R_h R_k)(S + R_k)}{S^{*2} + SR_k} \right)^{\tau(k)} \delta_h^{-(\tau(h)-t)}$$

Substituting (33) in (34) thus yields an inequality of the shape

$$(35) \qquad | F |_S (1 - Y) \leqslant E$$

with

$$(36) \qquad Y \leqslant \frac{\gamma e^{\Omega S^*}}{(\gamma - 1)^2} \left(\gamma^\sigma \sum_{h=0}^{\sigma-1} \frac{(\Omega S)^h}{h!} - \sum_{h=0}^{\sigma-1} \frac{(\Omega S^*)^h}{h!} \right) \prod_{k=1}^{n} \left(\frac{S^* (S + R_k)}{S^{*2} + SR_k} \right)^{\tau(k)},$$

and we require, in order that we obtain a meaningful result, that $Y < 1$. Firstly we simplify (36) as in theorem 1, and obtain on writing $\tau = (S^{*2} + SR)/S^* (S + R)$ that

$$(37) \qquad -\log Y \geqslant N \log \tau - (\sigma - 1) \log \gamma - \Omega (S^* + S) - 2 \log (\gamma/\gamma - 1)$$

We conclude that (seeing that the last term is insignificant) it suffices that $N > n(F, R, 0)$ in order that $-\log Y$ be positive. Moreover lemma 6 yields an inequality of the shape

$$(38) \qquad | b_{ks} | \leqslant | F |_S Z_{ks}$$

with

$$Z_{ks} \leqslant 2^{\sigma - s} D_k^{-1} \prod_{h=1}^{m} (\theta + \Omega_h)^{\rho(h) - \delta_{hk}} d_k^{-(\rho(k)-s)} .$$

Thus (35) together with (38) gives

$$| b_{ks} | \leqslant \frac{Z_{ks}}{1 - Y} E \tag{39}$$

which is of the shape we require. Then it remains to appropriately choose parameters and to make simplifications so as to obtain a result in simple shape.

For example select $\gamma = 10$, $\tau = 3{,}8$. Then we may choose $N = 2(\sigma - 1) + 5\Omega R$ and from (37) obtain that $1/(1 - Y) < 3$, (provided only that $\sigma > 1$). With this choice it suffices for the scaling of lemma 6, to choose $\theta \geqslant 2(\sigma - 1)/eR$ at (38). By now suppressing all details (that is, replacing all R_k, Ω_k, ... by R, Ω, ... respectively) and estimating E with the above choice for the parameters we get

$$E \leqslant \frac{1}{3} (5R)^N \sum_{h=1}^{n} \sum_{t=1}^{\tau(h)} N^{-1} \chi_{ht} \Delta_h^{-1} \delta_h^{-(\tau(h)-t)} \tag{40}$$

and

$$Z_{ks} \leqslant D_k^{-1} d_k^{-(\rho(k)-s)} (2/eR)^{\sigma-1} (2(\sigma - 1) + e\Omega R)_1^{\sigma-1} , \tag{41}$$

so certainly either

$$Z_{ks} \leqslant D_k^{-1} d_k^{-(\rho(k)-s)} \left\{ \max \left(8\Omega , \frac{2(\sigma - 1)}{R} \right) \right\}^{\sigma - 1} , \tag{42}$$

or, more tidily, though less sharply

$$Z_{ks} \leqslant D_k^{-1} d_k^{-(\rho(k)-s)} (2N/eR)^{\sigma-1} < D_k^{-1} d_k^{-(\rho(k)-s} (3N/4R)^{\sigma-1} . \tag{43}$$

We further recall that if $s = \rho(k)$ then (42) and (43) become respectively

$$Z_{k\rho(k)} \leqslant D_k^{-1} \left\{ \max \left(4\Omega , \frac{\sigma - 1}{R} \right) \right\}^{\sigma-1} , \quad Z_{k\rho(k)} \leqslant D_k^{-1} (N/eR)^{\sigma-1} .$$

These estimates yield the results of the theorem.

One can of course obtain alternative estimates more suitable to a particular application; in particular it would in practice be appropriate to select the parameters, and thus S^* and S, according to the relative sizes of $\sigma - 1$ and ΩR.

We have made a point of specially mentioning the simpler bounds for $| b_{k\rho(k)} |$ because in typical estimations in transcendence theory one has the

b_{ks}, or even the $b_{ks}/(s-1)!$ rational integers; thus as soon as $|b_{k\rho(k)}| < 1$ one has $b_{k\rho(k)} = 0$ and then *a fortiori* one has sequentially $b_{k\rho(k)\downarrow 1} = \ldots = b_{k1} = 0$ which eventually shows that F vanishes identically. Thus in this circumstance it suffices to have an estimate only for the leading coefficients of the polynomial coefficients.

In applications it is of course necessary to have good lower bounds for the D_k and the Δ_h. For some such estimates see Cijsouw and Tijdeman [5], lemmas 5 and 6.

One case is of sufficient interest to mention specifically:

COROLLARY. *If* $\alpha_1 = 1, \alpha_2 = 2, \ldots, \alpha_n = R$ *(so* $n = R$*) and* $\tau(h) = T\,(h = 1, \ldots, n)$, *so* $N = RT$, *then*

$$\left| \frac{F^{(t-1)}(h)}{(t-1)!} \right| \leqslant \chi_{ht} \leqslant \chi_h \leqslant \chi \quad (h = 1, \ldots, R; t = 1, \ldots, T)$$

and

$$N = RT \geqslant 2(\sigma - 1) + 5\Omega R,$$

implies that for $k = 1, \ldots, m$

$$|b_{k\rho(k)}| < D_k^{-1} (N/eR)^{\sigma-1} \sum_{h=1}^{R} \sum_{t=1}^{T} \chi_{ht} N^{-1} (5R)^N/((h-1)!\,(R-h)!\,)^T$$

$$< D_k^{-1} (N/eR)^{\sigma-1}\, 30^N \chi$$

Proof. Note only that $(h-1)!\,(R-h)! > 2^{-(R-1)}\,(R-1)! > (R/6)^R$; (by sharpening lemma 7 for this case one can improve the 30 to about 15).

5. FURTHER RESULTS

We consider some further applications of the method of this note. It is instructive to observe that the success of these applications depends, in effect, on forcing an analogy with the simplest case, that of exponential polynomials. The methods of Hayman [8] applies to a different class of functions, which does however intersect with the class considered here. For an example of this different method at work, see Voorhoeve, van der Poorten and Tijdeman [33]. In this context see also Voorhoeve and van der Poorten [32]; the ideas here however relate to the new method of Voorhoeve [31].

Continuing to use the notation of the previous sections, we observe that if in lemma 2 we take $t_\lambda = \lambda - 1$, $z_\lambda = 0$ and $g_i(z) = g(\omega_i z)$ where g is given by (14) then the ratio $\Delta_{\lambda,k}/\Delta$ of lemma 2 is given by

$$\Delta_{\lambda,k} / \Delta = D_{\lambda,k} / Dc_{\lambda-1} .$$

where D is the Vandermonde determinant of lemma 3. Then lemma 4 and lemma 5 allow us to estimate the number of zeros of functions F of the shape

$$(44) \qquad F(z) = \sum_{h=1}^{m} \sum_{t=1}^{\rho(h)} a_{ht} z^{t-1} g^{(t-1)}(\omega_h z)$$

in discs with centre the origin. Indeed, the analogue of (21) becomes

$$| F |_{S*} / | F |_S < \sum_{\lambda=1}^{\sigma} \left(\frac{S^*}{S}\right)^{\lambda-1} | c_{\lambda-1} |^{-1} \sum_{h=1}^{\sigma} \frac{(\Omega S^*)^{h-\lambda}}{(h-\lambda)!} | g |^{(h-1)} (\Omega S^*) ,$$

and the only important new addition is that one requires, if $g(z) = \sum \frac{c_n}{n!} z^n$, that $c_0 c_1 \dots c_{\sigma-1} \neq 0$.

An easy example is given by the class of functions

$$(45) \qquad g(z) = f_\mu(z) = \sum_{n=0}^{\infty} z^n / (\mu+1) \dots (\mu+n)$$

for μ in $\mathbf{C}$, μ not a negative integer. Here it is amusing to observe that one has

$$z f_\mu'(z) = \mu + (z-\mu) f_\mu(z) \text{ and hence } z^{t-1} f_\mu^{(t-1)}(z) = r_t(z;\mu) + q_t(z;\mu) f_\mu(z)$$

for $t = 1, 2 \dots$, where the polynomials r_t, q_t have degree respectively at most t-2 and t-1 in z. It follows that, with a slight change of notation, the function (44) can be taken to be of the shape

$$F(z) = \sum_{h=0}^{m} P_h(z) f_\mu(\omega_h z) ,$$

where the P_h are polynomials of degrees respectively at most $\rho(0)$, $\rho(1) - 1, \dots, \rho(m) - 1$ and $\rho(0) \geqslant \max_k \rho(k)$, $\omega_0 = 0$ (so $f_\mu(\omega_0 z) \equiv 1$), and we take $\sum_{h=0}^{m} \rho(h) = \sigma + 1$.

However one need not be as explicit as regards the Taylor coefficients of the given function g. For example consider a Weierstrass elliptic function $\wp$ with given fixed algebraic invariants. Then one easily shows that there is a point v such that

$$| \wp(v) | \leqslant c \text{ and } | \wp^{(\lambda-1)}(v) | \geqslant \sigma^{-c\sigma} , \quad \lambda = 1, \dots, \sigma$$

for some c depending only on $\wp$. It is then easy to conclude by the method we have described that if $\max_k | \omega_k | = \Omega \leqslant 1$ then a function $F \not\equiv 0$ of the shape

$$F(z) = \sum_{h=1}^{m} \sum_{t=1}^{\rho(h)} a_{ht} z^{t-1} \wp^{(t-1)}(\omega_h z + v)$$

cannot have more than $c' \sigma \log \sigma$ zeros in a disc $|z| \leqslant c''$, where c', c'' depend only on p. We are indebted for the above details to D. W. Masser (for a problem involving zeros of polynomials in several variables see his [13]).

To extend our method to a class of functions wider than that given by (44) is practical provided only that one can usefully estimate the determinants arising in lemma 2. This can certainly be done in the case

$$F(z) = \sum_{h=1}^{m} \sum_{t=1}^{\rho(h)} a_{ht} (\log z)^{t-1} z^{\alpha_h},$$

for details see van der Poorten [22]. A similar argument should allow one to deal with functions

$$\sum_{h=1}^{\sigma} b_h f_{\mu_h}(z),$$

where f_μ is given by (45); now lemma 5 allows one to consider rather surprising functions. There are further, rather isolated cases where one can deal with the determinants; for some examples, and further references see [21].

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