4. EXPONENTIAL POLYNOMIALS

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pattern is due to Tijdeman [26], whence see Balkema and Tijdeman [1]. For further details see the references cited in the papers mentioned above.

4. Exponential polynomials

We commence by making explicit some folklore the principles of which can be found in [16] and Tijdeman [26], and which is made explicit in another context in van der Poorten [20].

Lemma 5. *For some fixed positive integer* σ, *and some given function* g, *supposed holomorphic in the domain under consideration, denote by* J *the set of functions* G *of the shape*

$$G(z) = \sum_{k=1}^{\sigma} b_k g(\alpha_k z),$$

where $b_1, \ldots, b_\sigma; \alpha_1, \ldots, \alpha_\sigma$ *are complex numbers. Then, for all sets of non-negative integers* $\rho(1), \ldots, \rho(m)$ with *sum* $\sum_{h=1}^{m} \rho(h) = \sigma$ *(and all positive integers* m *such that* $1 \leqslant m \leqslant \sigma$), *for each function* F *of the shape*

$$F(z) = \sum_{h=1}^{m} \sum_{t=1}^{\rho(h)} a_{ht} z^{t-1} g^{(t-1)}(\omega_h z),$$

the a_{ht} *complex constants, there is a sequence of functions in* J *converging uniformly to* F *in compact sets.*

Proof. The lemma depends upon noticing that functions of the shape F are actually, in a sense, particular cases of, rather than generalisations of functions of the shape G. Indeed, reindex so that G appears as

$$G(z) = \sum_{h=1}^{m} \sum_{t=1}^{\rho(h)} b_{ht} g(\omega_{ht} z), \tag{17}$$

and choose the coefficients b_{ht} as functions of $\omega_{11}, \ldots, \omega_{m\rho(m)}$ (so of $\alpha_1, \ldots, \alpha_\sigma$) so that for each $h = 1, \ldots, m$

$$\sum_{t=1}^{\rho(h)} b_{ht} g(\omega_{ht} z) = \sum_{t=1}^{\rho(h)} a_{ht} \frac{(t-1)!}{2\pi i} \int_C g(\gamma z) \prod_{s=1}^{t} (\gamma - \omega_{hs})^{-1} d\gamma, \tag{18}$$

where the closed contour C contains all the ω_{ht} but excludes any singularities of g. Clearly there exists a sequence of σ-tuples $(\omega_{11}, \ldots, \omega_{m\rho(m)})$ which converges to $(\omega_1, \ldots, \omega_1; \omega_2, \ldots, \omega_m)$ componentwise, and in the limit, (18) shows that (17) becomes $F(z)$.

I am indebted to D. W. Masser for any felicities in the terminology used in the lemma.

The following theorem is a result due to Tijdeman [26], [27]; see also Waldschmidt [36] p. 164-174. A history of the problem can be found in the above mentioned works.

DEFINITION. Let $\rho(1), \ldots, \rho(m)$ be non-negative integers with sum $\sum \rho(h) = \sigma$, let a_{ht} $(h=1, \ldots, m;\ t=1, \ldots, \rho(h))$ be complex numbers which do not all vanish, and let $\omega_1, \ldots, \omega_m$ be distinct complex numbers. Then a function of the shape

$$F(z) = \sum_{h=1}^{m} \sum_{t=1}^{\rho(h)} a_{ht}\, z^{t-1} e^{\omega_h z} \tag{19}$$

is called an *exponential polynomial* of *degree* σ, with *frequencies* ω_h and *coefficients* a_{ht}.

THEOREM 1. *The number of zeros* $n(F, R, z_0)$ *of an exponential polynomial* F *of degree* σ *and with frequencies* $\omega_1, \ldots, \omega_m$ *satisfying* $\max_h |\omega_h| \leqslant \Omega$, *in a disc of centre* z_0 *and radius* R, *is less than*

$$\frac{\log \gamma}{\log \tau}(\sigma - 1) + \frac{(\gamma\tau - 1)(\gamma + 1)}{\gamma(\gamma - \tau)\log \tau}\,\Omega R + \frac{1}{\log \tau\,(\gamma - 1)}, \tag{20}$$

for all $\gamma > \tau > 1$.

Proof. We consider the exponential sum

$$G(z) = \sum_{k=1}^{\sigma} b_k e^{\omega_k z}, \quad \max_k |\omega_k| \leqslant \Omega,$$

and suppose that $z_0 = 0$. In lemma 2 take $g_k(z) = e^{\omega_k z}$ and t_j, z_j such that $t_\lambda = \lambda - 1$, $z_\lambda = 0$ $(\lambda = 1, \ldots, \sigma)$ and observe that the determinant Δ of lemma 2 now coincides with the Vandermonde determinant D of lemmas 3 and 4. Then from lemmas 2 and 4 we obtain

$$|G|_{S^*}/|G|_S \leqslant \sum_{\lambda=1}^{\sigma} \left(\frac{S^*}{S}\right)^{\lambda-1} \sum_{h=1}^{\sigma} \frac{(\Omega S^*)^{h-\lambda}}{(h-\lambda)!}\, e^{\Omega S^*}. \tag{21}$$

We observe that the information (21) is independent of the coefficients of G and independent of $\min_{h \neq k} |\omega_k - \omega_h|$; furthermore, under a translation only the coefficients of G change, so (21) is valid for all centres z_0. So by lemma 5 we have, writing $S^*/S = \gamma > 1$,

$$\begin{aligned} |F|_{S^*}/|F|_S &\leqslant e^{\Omega S^*} \sum_{h=0}^{\sigma-1} \frac{(\Omega S^*)^h}{h!} \sum_{\lambda=1}^{\sigma-h} \gamma^{\lambda-1} \\ &= \frac{e^{\Omega S^*}}{\gamma - 1}\left(\gamma^\sigma \sum_{h=0}^{\sigma-1} \frac{(\Omega S)^h}{h!} - \sum_{h=0}^{\sigma-1} \frac{(\Omega S^*)^h}{h!}\right) \end{aligned} \tag{22}$$

It follows that, with the gain of some tidiness, but the loss of some precision, $\left(\text{and noticing that} \quad -\log\left(1-\frac{1}{\gamma}\right) < \frac{1}{\gamma-1}\right)$,

$$\log |F|_{S^*}/|F|_S < (\sigma-1)\log\gamma + \frac{1}{\gamma-1} + \Omega(S^*+S)\,.$$

Finally let $\tau = (S^{*2}+SR)/S^*(S+R) > 1$; then lemma 1 implies that

$$(23) \qquad n(F, R, z_0) < \frac{1}{\log\tau}\left\{(\sigma-1)\log\gamma + \frac{1}{\gamma-1} + \Omega R\,\frac{(\gamma\tau-1)(\gamma+1)}{\gamma(\gamma-\tau)}\right\}$$

as asserted.

COROLLARY 1. *If* $\sigma = 1$ *then* $n(F, R, z_0) = 0$. *For* $\sigma > 1$ *we have*

(24) $\quad n(F, R, z_0) < 3(\sigma-1) + 4\Omega R\,,$ (Tijdeman [27]

or

(25) $\quad n(F, R, z_0) < 2(\sigma-1) + 5\Omega R\,,$ (Waldschmidt [36])

or

(26) $\quad n(F, R, z_0) < 4(\sigma-1) + 3\Omega R\,.$

Proof. The first remark is trivial; we require $\sigma > 1$ in order to assimilate the term $1/(\gamma-1)$ in (20). To obtain (24) choose, say, $\tau = 3.5$, $\gamma = 30$, and for (25) $\tau = 3.5$, $\gamma = 10$, whilst for (26) $\tau = 3.5$, $\gamma = 110$. Parameters were calculated on the *HP* 65 belonging to John Conway, for whom see Knuth [9].

Notwithstanding the apparent precision of our method, (20) gives quite inadequate results in the asymptotic cases. For example, we know from results of Pólya [15] and Dickson [6] that $\lim_{R\to\infty} n(F, R)/R < \Omega$, but (22) does no better than $\lim_{R\to\infty} n(F, R)/R \leqslant e\Omega$. At the opposite extreme, " the local valency problem ", M. Voorhoeve has shown, using an idea of Hayman [8], that if $\sigma \geqslant 4$ then $R \leqslant 1/8\,\Omega$ implies $n(F, R) \leqslant \sigma - 1$, but nothing like this precision is available from (20); incidentally, because F has σ coefficients, it is clear that in every disc, no matter how small, one may have $n(F, R) \geqslant \sigma - 1$.

Although theorem 1 is more than adequate for applications to transcendence arguments, one can do better; for example Voorhoeve [31] has shown that $n(F, R, z_0) \leqslant 2(\sigma-1) + \frac{4}{\pi}\Omega R$ by a quite different argument. Actually because the result in the exponential polynomial case is independent of centre z_0, lemma 1 is quite crude (note the " extreme case ") because it assumes that the zeros accumulate at a point near the edge of the disc.

We now turn to a generalisation of theorem 1 wherein we show that an exponential polynomial cannot be small at too many points. This result largely includes earlier similar results of Mahler [10], Tijdeman [29] and Cijsouw and Tijdeman [5]. As we shall see, theorem 2 actually contains theorem 1 as a special case.

We shall retain, without further explanation, the notations introduced in the lemmas above. In preparation for the proof of the theorem we require two lemmas.

LEMMA 6. *Let*

$$F(z) = \sum_{k=1}^{m} \sum_{s=1}^{\rho(k)} \frac{b_{ks}}{(s-1)!} z^{s-1} e^{\omega_k z}$$

be an exponential polynomial of degree at most $\sigma = \sum_{k=1}^{m} \rho(k)$.

Write

$$D_k = \left| \prod_{\substack{h=1 \\ h \neq k}}^{m} (\omega_k - \omega_h)^{\rho(h)} \right|, \quad \Omega_k = |\omega_k|, \quad d_k = \min_{h \neq k} |\omega_k - \omega_h|$$

$(k = 1, \ldots, m)$

Denote by θ *a real number such that* $(S\theta)^{\sigma-1} \geqslant (\sigma-1)!$ *and by* S *a real number we shall suitably determine below. Then*

$$|b_{ks}| \leqslant 2^{\sigma-s} D_k^{-1} \prod_{h=1}^{m} (\theta + \Omega h)^{\rho(h)-\delta_{hk}} d_k^{-(\rho(k)-s)} |F|_S$$

and in particular

$$|b_{k\rho(k)}| \leqslant D_k^{-1} \prod_{h=1}^{m} (\theta + \Omega h)^{\rho(h)-\delta_{hk}} |F|_S \quad (k = 1, \ldots m; s = 1, \ldots, \rho(k))$$

Proof. Notice (compare lemma 2) that if $\Delta_{\lambda,ks}$ is the cofactor of the typical element in the $\sigma \times \sigma$ determinant

$$\Delta = \left| \binom{\lambda-1}{s-1} \omega_k^{\lambda-s} \right|_{ks,\lambda}$$

(here rows are indexed by the pairs (k, s) and columns by λ) then plainly

$$b_{ks} = \sum_{\lambda=1}^{\sigma} \left\{ \frac{(\lambda-1)!}{2\pi i} \int_{|\zeta|=S} \frac{\Delta_{\lambda,ks}}{\Delta} F(\zeta) \frac{d\zeta}{\zeta^\lambda} \right\},$$

so

$$|b_{ks}| \leqslant |F|_S \sum_{\lambda=1}^{\sigma} \left| \frac{\Delta_{\lambda,ks}}{\Delta} \right| \frac{(\lambda-1)!}{S^{\lambda-1}}. \tag{27}$$

But

$$\sum_{\lambda=1}^{\sigma} \frac{(\lambda-1)!}{(\lambda-t)!}\, \omega_h^{\lambda-t}\, \frac{\Delta_{\lambda,ks}}{\Delta} = \begin{cases} (s-1)! & (h,t) = (k,s) \\ 0 & (h,t) \neq (k,s) \end{cases}$$

so $\Delta_{\lambda,ks}/\Delta$ is exactly the coefficient of $z^{\lambda-1}$ in the polynomial $P_{ks}(z)$ of degree at most $\sigma - 1$ defined by the σ conditions

$$P_{ks}^{(t-1)}(\omega_h) = \begin{cases} (s-1)! & (h,t) = (k,s) \\ 0 & (h,t) \neq (k,s) \end{cases}$$

We now make a change of scale whereby we replace z in $F(z)$ by $z\,\theta$; this is tantamount to replacing each ω_k by ω_k/θ whilst each b_{ks} become b_{ks}/θ^{s-1}. We arrange that $(S\theta)^{\sigma-1} \geqslant (\sigma-1)!$. Then (27) implies that it suffices to find an upper bound for the sum of absolute values of the coefficients of $P_{ks}(z)$. There is a useful stratagem whereby one obtains such a bound, dependent on the observation that if a polynomial $P(z)$ has non-negative real zeros then $|P(-1)|$ is the sum of the absolute values of its coefficients. For formal details of the required generalisation of this remark see van der Poorten [17], lemma 2.

One confirms readily that the polynomial $P_{ks}(z)$ is given by the integral

$$(28) \qquad P_{ks}(z) = -\frac{1}{2\pi i} \int_{C_k} \frac{(\zeta-\omega_k)^{s-1}}{\zeta - z} \prod_{h=1}^{m} \left(\frac{z-\omega_h}{\zeta-\omega_h}\right)^{\rho(h)} d\zeta$$

where C_k is a suitable contour about ω_k excluding the other ω_h and formally excluding z; in fact (28) is just a special case of an integral form of the Hermite interpolation formula.

So

$$P_{ks}(z) = D_k^{-1} \frac{(z-\omega_k)^{\rho(k)-1}}{(\rho(k)-s)!} \left(\frac{\partial}{\partial \zeta}\right)^{\rho(k)-s} \left\{ \prod_{\substack{h=1 \\ h\neq k}}^{m} \left\{ \frac{\dfrac{z-\omega_k}{\zeta-\omega_k}}{\dfrac{\zeta-\omega_k}{\omega_h-\omega_k} - 1} \right\}^{\rho(h)} \cdot \frac{1}{\zeta - z} \right\}_{\zeta=\omega_k}$$

whence

$$(29) \quad |P_{ks}(-1)| \leqslant D_k^{-1}\, 2^{\sigma-s} \prod_{h=1}^{m} (1+\Omega_h)^{\rho(h)-\delta_{hk}}\, d_k^{-(\rho(k)-s)}$$

But in the estimate (29) we have estimated the numerator as if $P_{ks}(z)$ were a sum of polynomials with non-negative real zeros $\Omega_1, \ldots, \Omega_m$. Hence (29) gives an upper bound for the sum of the absolute values of the coefficients of $P_{ks}(z)$. Recalling the scaling we are assuming, (27) implies that

$$|b_{ks}| \leqslant 2^{\sigma-s} D_k^{-1} \prod_{h=1}^{m} (\theta+\Omega_h)^{\rho(h)-\delta_{hk}}\, d_k^{-(\rho(k)-s)}\, |F|_S$$

In the special case $s = \rho(k)$ we have very simply that

$$P_{k\rho(k)}(z) = \prod_{h=1}^{m} \left(\frac{z-\omega_h}{\omega_k-\omega_h}\right)^{\rho(h)-\delta_{hk}}$$

and the estimate for $b_{k\rho(k)}$ follows immediately from (27) and the argument outlined above.

LEMMA 7. *Let* $\alpha_1, \ldots, \alpha_n$ *be distinct points such that*

$$\left|\frac{F^{(t-1)}(\alpha_h)}{(t-1)!}\right| \leqslant \chi_{ht} \leqslant \chi_h \leqslant \chi$$

and

$$\Delta_h = \left| \prod_{\substack{k=1\\k\neq h}}^{m} (\alpha_h-\alpha_k)^{\tau(k)} \right|, \quad |\alpha_h| = R_h \leqslant R, \quad \min_{k\neq h} |\alpha_h - \alpha_k| = \delta_h$$

$(h=1, \ldots, n;\ t=1, \ldots, \tau(h))$ *where* $\tau(1), \ldots, \tau(n)$ *are positive integers with sum* $\sum_{h=1}^{n} \tau(h) = N$.

Let S^*, S *be real numbers satisfying* $S^* > S > 0$ *and* $S^* \geqslant R > 0$.
Then.

$$|F|_S \leqslant \frac{S^*}{S^*-S} \prod_{k=1}^{n} \left(\frac{S^*(S+R_k)}{S^{*2}+SR_k}\right)^{\tau(k)} |F|_{S^*} +$$

$$+ \sum_{h=1}^{n} \sum_{t=1}^{\tau(h)} \chi_{ht} N^{-1} 3^{N-1} \Delta_h^{-1} \prod_{k=1}^{n} \left(\frac{(S^*+R_hR_k)(S+R_k)}{S^{*2}+SR_k}\right)^{\tau(k)} \delta_h^{-(\tau(h)-t)}$$

Proof. By an integral form of the Hermite interpolation formula we have

$$\frac{1}{2\pi i}\int_{|\zeta|=S} F(\zeta) \prod_{k=1}^{n} \left(\frac{S^{*2}-\zeta\bar{\alpha}_k}{S^*(\zeta-\alpha_k)}\right)^{\tau(k)} \frac{d\zeta}{\zeta-z}$$

$$= F(z) \prod_{k=1}^{n} \left(\frac{S^{*2}-z\bar{\alpha}_k}{S^*(z-\alpha_k)}\right)^{\tau(k)} +$$

$$+ \frac{1}{2\pi i} \sum_{h=1}^{n} \sum_{t=1}^{\tau(h)} \frac{F^{(t-1)}(\alpha_h)}{(t-1)!} \int_{C_h} (\zeta-\alpha_h)^{t-1} \prod_{k=1}^{n} \left(\frac{(S^{*2}-\zeta\bar{\alpha}_k}{(S^*(\zeta-\alpha_k)}\right)^{\tau(k)} \frac{d\zeta}{\zeta-z}$$

as can be seen directly by the Cauchy residue theorem; here C_h is a suitable contour about α_h excluding the other α_k and formally excluding z. By the argument detailed in lemma 2 we have for $|z| = S$.

$$\left|\prod_{k=1}^{n}\left(\frac{S^{*2}-z\bar{\alpha}_k}{S^*(z-\alpha_k)}\right)^{\tau(k)}\right| \geqslant \prod_{k=1}^{n}\left(\frac{(S^{*2}+SR_k}{(S^*(S+R_k)}\right)^{\tau(k)} \tag{30}$$

and

$$(31) \qquad \left| \frac{1}{2\pi i} \int_{|\zeta| = S^*} F(\zeta) \prod_{k=1}^{n} \left(\frac{S^{*2} - \zeta \bar{\alpha}_k}{S^* (\zeta - \alpha_k)} \right)^{\tau(k)} \frac{d\zeta}{\zeta - z} \right| \leqslant | F |_{S^*} \frac{S^*}{S^* - S}$$

We select z such that $| F(z) | = | F |_s$ whence we may suppose that z is not near any α_h. Then by explicit evaluation similar to that in lemma 6 we obtain.

$$(32) \qquad \left| \frac{1}{2\pi i} \int_{C_h} (\zeta - \alpha_h)^{t-1} \prod_{k=1}^{n} \left(\frac{S^{*2} - \zeta \bar{\alpha}_k}{S^* (\zeta - \alpha_k)} \right)^{\tau(k)} \frac{d\zeta}{\zeta - z} \right|$$

$$\leqslant N^{-1} \, 3^{N-1} \, \Delta_h^{-1} \prod_{k=1}^{n} \left(\frac{S^{*2} + R_h R_k}{S^*} \right)^{\tau(k)} \delta_h^{-(\tau(h) - t)} .$$

The three inequalities (30), (31) and (32) together with the integral interpolation formula now readily yield the lemma.

THEOREM 2. *Let*

$$F(z) = \sum_{k=1}^{m} \sum_{s=1}^{\rho(k)} \frac{b_{ks}}{(s-1)!} z^{s-1} e^{\omega_k z}$$

be an exponential polynomial of degree at most $\sigma = \sum_{k=1}^{m} \rho(k) \quad (>1)$

with

$$D_k = | \prod_{\substack{h=1 \\ h \neq k}}^{m} (\omega_k - \omega_h)^{\rho(h)} | , \quad | \omega_k | = \Omega_k \leqslant \Omega ,$$

$$d_k = \min_{h \neq k} | \omega_k - \omega_h | \quad (k = 1, \ldots, m)$$

Further let $\alpha_1, \ldots, \alpha_n$ *be distinct points such that*

$$\Delta_h = | \prod_{\substack{k=1 \\ k \neq h}}^{n} (\alpha_h - \alpha_k)^{\tau(k)} | , \quad | \alpha_h | = R_h \leqslant R , \quad \min_{k \neq h} | \alpha_h - \alpha_k | = \delta_h .$$

and

$$\left| \frac{F^{(t-1)}(\alpha_h)}{(t-1)!} \right| \leqslant \chi_{ht} \leqslant \chi_h \leqslant \chi$$

$(h = 1, \ldots n;\ t = 1, \ldots, \tau(h))$ *where* $\tau(1), \ldots, \tau(n)$ *are positive integers with sum* $\sum_{h=1}^{n} \tau(h) = N.$

Then if

$$N \geqslant 2(\sigma - 1) + 5\Omega R$$

we have

$$| b_{ks} | < D_k^{-1} d_k^{-(\rho(k)-s)} (3N/4R)^{\sigma-1} \sum_{h=1}^{n} \sum_{t=1}^{\tau(h)} \chi_{ht} N^{-1} \Delta_h^{-1} \delta_h^{-(\tau(h)-t)} (5R)^N .$$

$$\leqslant D_k^{-1} d_k^{-(\rho(k)-s)} (3N/4R)^{\sigma-1} \Delta^{-1} (5R)^N \chi ,$$

(*where* $\Delta = \min\limits_{(h,t)} \Delta_h \delta_h^{\tau(h)-t}$). $\quad (k=1, \ldots, m;\ s=1, \ldots, \rho(k))$

In particular for $k = 1, \ldots, m$

$$| b_{k\rho(k)} | < D_k^{-1} (N/eR)^{\sigma-1} \sum_{h=1}^{n} \sum_{t=1}^{\tau(h)} \chi_{ht} N^{-1} \Delta_h^{-1} \delta_h^{-(\tau(h)-t)} (5R)^N$$

Proof. In the proof of theorem 1 we showed at (22) that

$$(33) \qquad | F |_{S^*} \leqslant | F |_S \frac{e^{\Omega S^*}}{\gamma - 1} \left((\gamma^\sigma \sum_{h=0}^{\sigma-1} \frac{(\Omega S)^h}{h!} - \sum_{h=0}^{\sigma-1} \frac{(\Omega S^*)^h}{h!} \right)$$

whilst lemma 7 gives an inequality

$$(34) \qquad | F |_S \leqslant | F |_{S^*} \frac{\gamma}{\gamma - 1} \prod_{k=1}^{n} \left(\frac{S^* (S + R_k)}{S^{*2} + SR_k} \right)^{\tau(k)} + E$$

with

$$E \leqslant \sum_{h=1}^{n} \sum_{t=1}^{\tau(h)} \chi_{ht} N^{-1} 3^{N-1} \Delta_h^{-1} \prod_{k=1}^{n} \left(\frac{(S^{*2} + R_h R_k)(S + R_k)}{S^{*2} + SR_k} \right)^{\tau(k)} \delta_h^{-(\tau(h)-t)}$$

Substituting (33) in (34) thus yields an inequality of the shape

$$(35) \qquad | F |_S (1 - Y) \leqslant E$$

with

$$(36) \qquad Y \leqslant \frac{\gamma e^{\Omega S^*}}{(\gamma - 1)^2} \left(\gamma^\sigma \sum_{h=0}^{\sigma-1} \frac{(\Omega S)^h}{h!} - \sum_{h=0}^{\sigma-1} \frac{(\Omega S^*)^h}{h!} \right) \prod_{k=1}^{n} \left(\frac{S^* (S + R_k)}{S^{*2} + SR_k} \right)^{\tau(k)},$$

and we require, in order that we obtain a meaningful result, that $Y < 1$. Firstly we simplify (36) as in theorem 1, and obtain on writing $\tau = (S^{*2} + SR)/S^* (S + R)$ that

$$(37) \qquad -\log Y \geqslant N \log \tau - (\sigma - 1) \log \gamma - \Omega (S^* + S) - 2 \log (\gamma/\gamma - 1)$$

We conclude that (seeing that the last term is insignificant) it suffices that $N > n (F, R, 0)$ in order that $-\log Y$ be positive. Moreover lemma 6 yields an inequality of the shape

$$(38) \qquad | b_{ks} | \leqslant | F |_S Z_{ks}$$

with

$$Z_{ks} \leqslant 2^{\sigma - s} D_k^{-1} \prod_{h=1}^{m} (\theta + \Omega_h)^{\rho(h) - \delta_{hk}} d_k^{-(\rho(k) - s)} .$$

Thus (35) together with (38) gives

$$(39) \qquad | b_{ks} | \leqslant \frac{Z_{ks}}{1 - Y} E$$

which is of the shape we require. Then it remains to appropriately choose parameters and to make simplifications so as to obtain a result in simple shape.

For example select $\gamma = 10$, $\tau = 3{,}8$. Then we may choose $N = 2(\sigma - 1) + 5\Omega R$ and from (37) obtain that $1/(1 - Y) < 3$, (provided only that $\sigma > 1$). With this choice it suffices for the scaling of lemma 6, to choose $\theta \geqslant 2(\sigma - 1)/eR$ at (38). By now suppressing all details (that is, replacing all $R_k, \Omega_k, \ldots$ by $R, \Omega, \ldots$ respectively) and estimating E with the above choice for the parameters we get

$$(40) \qquad E \leqslant \frac{1}{3} (5R)^N \sum_{h=1}^{n} \sum_{t=1}^{\tau(h)} N^{-1} \chi_{ht} \Delta_h^{-1} \delta_h^{-(\tau(h) - t)}$$

and

$$(41) \quad Z_{ks} \leqslant D_k^{-1} d_k^{-(\rho(k) - s)} (2/eR)^{\sigma - 1} (2(\sigma - 1) + e\Omega R)_1^{\sigma - 1} ,$$

so certainly either

$$(42) \qquad Z_{ks} \leqslant D_k^{-1} d_k^{-(\rho(k) - s)} \left\{ \max \left(8\Omega , \frac{2(\sigma - 1)}{R} \right) \right\}^{\sigma - 1} ,$$

or, more tidily, though less sharply

$$(43) \quad Z_{ks} \leqslant D_k^{-1} d_k^{-(\rho(k) - s)} (2N/eR)^{\sigma - 1} < D_k^{-1} d_k^{-(\rho(k) - s} (3N/4R)^{\sigma - 1} .$$

We further recall that if $s = \rho(k)$ then (42) and (43) become respectively

$$Z_{k\rho(k)} \leqslant D_k^{-1} \left\{ \max \left(4\Omega , \frac{\sigma - 1}{R} \right) \right\}^{\sigma - 1} , \quad Z_{k\rho(k)} \leqslant D_k^{-1} (N/eR)^{\sigma - 1} .$$

These estimates yield the results of the theorem.

One can of course obtain alternative estimates more suitable to a particular application; in particular it would in practice be appropriate to select the parameters, and thus S^* and S, according to the relative sizes of $\sigma - 1$ and ΩR.

We have made a point of specially mentioning the simpler bounds for $| b_{k\rho(k)} |$ because in typical estimations in transcendence theory one has the

b_{ks}, or even the $b_{ks}/(s-1)!$ rational integers; thus as soon as $|b_{k\rho(k)}| < 1$ one has $b_{k\rho(k)} = 0$ and then *a fortiori* one has sequentially $b_{k\rho(k)\downarrow 1} = \ldots = b_{k1} = 0$ which eventually shows that F vanishes identically. Thus in this circumstance it suffices to have an estimate only for the leading coefficients of the polynomial coefficients.

In applications it is of course necessary to have good lower bounds for the D_k and the Δ_h. For some such estimates see Cijsouw and Tijdeman [5], lemmas 5 and 6.

One case is of sufficient interest to mention specifically:

COROLLARY. *If* $\alpha_1 = 1, \alpha_2 = 2, \ldots, \alpha_n = R$ *(so* $n = R$*) and* $\tau(h) = T\,(h=1, \ldots, n)$, *so* $N = RT$, *then*

$$\left| \frac{F^{(t-1)}(h)}{(t-1)!} \right| \leqslant \chi_{ht} \leqslant \chi_h \leqslant \chi \quad (h = 1, \ldots, R;\, t = 1, \ldots, T)$$

and

$$N = RT \geqslant 2(\sigma - 1) + 5\Omega R\,,$$

implies that for $k = 1, \ldots, m$

$$|b_{k\rho(k)}| < D_k^{-1}(N/eR)^{\sigma-1} \sum_{h=1}^{R} \sum_{t=1}^{T} \chi_{ht} N^{-1} (5R)^N/((h-1)!\,(R-h)!\,)^T$$
$$< D_k^{-1}(N/eR)^{\sigma-1}\, 30^N \chi$$

Proof. Note only that $(h-1)!\,(R-h)! > 2^{-(R-1)}(R-1)! > (R/6)^R$; (by sharpening lemma 7 for this case one can improve the 30 to about 15).

5. Further results

We consider some further applications of the method of this note. It is instructive to observe that the success of these applications depends, in effect, on forcing an analogy with the simplest case, that of exponential polynomials. The methods of Hayman [8] applies to a different class of functions, which does however intersect with the class considered here. For an example of this different method at work, see Voorhoeve, van der Poorten and Tijdeman [33]. In this context see also Voorhoeve and van der Poorten [32]; the ideas here however relate to the new method of Voorhoeve [31].

Continuing to use the notation of the previous sections, we observe that if in lemma 2 we take $t_\lambda = \lambda - 1$, $z_\lambda = 0$ and $g_i(z) = g(\omega_i z)$ where g is given by (14) then the ratio $\Delta_{\lambda,k}/\Delta$ of lemma 2 is given by