5. Further results

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b_{ks}, or even the $b_{ks}/(s-1)!$ rational integers; thus as soon as $|b_{k\rho(k)}| < 1$ one has $b_{k\rho(k)} = 0$ and then *a fortiori* one has sequentially $b_{k\rho(k)\downarrow 1} = \ldots = b_{k1} = 0$ which eventually shows that F vanishes identically. Thus in this circumstance it suffices to have an estimate only for the leading coefficients of the polynomial coefficients.

In applications it is of course necessary to have good lower bounds for the D_k and the Δ_h. For some such estimates see Cijsouw and Tijdeman [5], lemmas 5 and 6.

One case is of sufficient interest to mention specifically:

COROLLARY. *If* $\alpha_1 = 1, \alpha_2 = 2, \ldots, \alpha_n = R$ *(so* $n = R$*) and* $\tau(h) = T\,(h=1, \ldots, n)$*, so* $N = RT$*, then*

$$\left|\frac{F^{(t-1)}(h)}{(t-1)!}\right| \leqslant \chi_{ht} \leqslant \chi_h \leqslant \chi \quad (h = 1, \ldots, R; t = 1, \ldots, T)$$

and

$$N = RT \geqslant 2(\sigma - 1) + 5\Omega R\,,$$

implies that for $k = 1, \ldots, m$

$$|b_{k\rho(k)}| < D_k^{-1}(N/eR)^{\sigma-1} \sum_{h=1}^{R} \sum_{t=1}^{T} \chi_{ht} N^{-1} (5R)^N/((h-1)!\,(R-h)!\,)^T$$
$$< D_k^{-1}(N/eR)^{\sigma-1}\, 30^N \chi$$

Proof. Note only that $(h-1)!\,(R-h)! > 2^{-(R-1)}(R-1)! > (R/6)^R$; (by sharpening lemma 7 for this case one can improve the 30 to about 15).

5. Further results

We consider some further applications of the method of this note. It is instructive to observe that the success of these applications depends, in effect, on forcing an analogy with the simplest case, that of exponential polynomials. The methods of Hayman [8] applies to a different class of functions, which does however intersect with the class considered here. For an example of this different method at work, see Voorhoeve, van der Poorten and Tijdeman [33]. In this context see also Voorhoeve and van der Poorten [32]; the ideas here however relate to the new method of Voorhoeve [31].

Continuing to use the notation of the previous sections, we observe that if in lemma 2 we take $t_\lambda = \lambda - 1$, $z_\lambda = 0$ and $g_i(z) = g(\omega_i z)$ where g is given by (14) then the ratio $\Delta_{\lambda, k}/\Delta$ of lemma 2 is given by

$$\Delta_{\lambda,k} / \Delta = D_{\lambda,k} / D c_{\lambda-1} .$$

where D is the Vandermonde determinant of lemma 3. Then lemma 4 and lemma 5 allow us to estimate the number of zeros of functions F of the shape

(44) $$F(z) = \sum_{h=1}^{m} \sum_{t=1}^{\rho(h)} a_{ht} z^{t-1} g^{(t-1)}(\omega_h z)$$

in discs with centre the origin. Indeed, the analogue of (21) becomes

$$| F |_{S*} / | F |_S < \sum_{\lambda=1}^{\sigma} \left(\frac{S^*}{S}\right)^{\lambda-1} | c_{\lambda-1} |^{-1} \sum_{h=1}^{\sigma} \frac{(\Omega S^*)^{h-\lambda}}{(h-\lambda)!} | g |^{(h-1)} (\Omega S^*) ,$$

and the only important new addition is that one requires, if $g(z) = \sum \frac{c_n}{n!} z^n$, that $c_0 c_1 \ldots c_{\sigma-1} \neq 0$.

An easy example is given by the class of functions

(45) $$g(z) = f_\mu(z) = \sum_{n=0}^{\infty} z^n / (\mu+1) \ldots (\mu+n)$$

for μ in $\mathbf{C}$, μ not a negative integer. Here it is amusing to observe that one has

$$z f_\mu'(z) = \mu + (z-\mu) f_\mu(z) \text{ and hence } z^{t-1} f_\mu^{(t-1)}(z) = r_t(z;\mu) + q_t(z;\mu) f_\mu(z)$$

for $t = 1, 2 \ldots$, where the polynomials r_t, q_t have degree respectively at most t-2 and t-1 in z. It follows that, with a slight change of notation, the function (44) can be taken to be of the shape

$$F(z) = \sum_{h=0}^{m} P_h(z) f_\mu(\omega_h z) ,$$

where the P_h are polynomials of degrees respectively at most $\rho(0)$, $\rho(1) - 1, \ldots, \rho(m) - 1$ and $\rho(0) \geqslant \max_k \rho(k)$, $\omega_0 = 0$ (so $f_\mu(\omega_0 z) \equiv 1$), and we take $\sum_{h=0}^{m} \rho(h) = \sigma + 1$.

However one need not be as explicit as regards the Taylor coefficients of the given function g. For example consider a Weierstrass elliptic function $\wp$ with given fixed algebraic invariants. Then one easily shows that there is a point v such that

$$| \wp(v) | \leqslant c \text{ and } | \wp^{(\lambda-1)}(v) | \geqslant \sigma^{-c\sigma} , \quad \lambda = 1, \ldots, \sigma$$

for some c depending only on $\wp$. It is then easy to conclude by the method we have described that if $\max_k |\omega_k| = \Omega \leqslant 1$ then a function $F \not\equiv 0$ of the shape

$$F(z) = \sum_{h=1}^{m} \sum_{t=1}^{\rho(h)} a_{ht} z^{t-1} \wp^{(t-1)}(\omega_h z + v)$$

cannot have more than $c' \sigma \log \sigma$ zeros in a disc $|z| \leqslant c''$, where c', c'' depend only on p. We are indebted for the above details to D. W. Masser (for a problem involving zeros of polynomials in several variables see his [13]).

To extend our method to a class of functions wider than that given by (44) is practical provided only that one can usefully estimate the determinants arising in lemma 2. This can certainly be done in the case

$$F(z) = \sum_{h=1}^{m} \sum_{t=1}^{\rho(h)} a_{ht} (\log z)^{t-1} z^{\alpha_h},$$

for details see van der Poorten [22]. A similar argument should allow one to deal with functions

$$\sum_{h=1}^{\sigma} b_h f_{\mu_h}(z),$$

where f_μ is given by (45); now lemma 5 allows one to consider rather surprising functions. There are further, rather isolated cases where one can deal with the determinants; for some examples, and further references see [21].

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