Introduction

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STABILITY OF PROJECTIVE VARIETIES [1])

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INTRODUCTION

The most direct approach to the construction of moduli spaces of algebraic varieties is via the theory of invariants: one describes the varieties by some sort of numerical projective data, canonically up to the action of some algebraic group, and then seeks to make these numbers canonical by applying invariant polynomials to the data, or equivalently by forming a quotient of the data by the group action. The main difficulty in this approach is to prove that "enough invariants exist": their values on the projective data must distinguish non-isomorphic varieties.

Take as an example the moduli space $\mathscr{M}_g$ of curves of genus $g \geqq 2$ over some algebraically closed field k. Given C, such a curve, we obtain by choosing a basis B of $\Gamma\left(C, (\Omega_c^1)^{\otimes l}\right)$, an embedding $\Phi\colon C \to \mathbf{P}^{(2l-1)(g-1)-1}$

[1]) Lectures given at the "Institut des Hautes Etudes Scientifiques", Bures-sur-Yvette (France), March-April 1976, under the sponsorship of the International Mathematical Union. Notes by Ian Morrison.

$= \mathbf{P}^N$. Let F be the Chow form of $\Phi(C)$ (cf. 1.16). Changing the basis B subjects $\Phi(C)$ to a projective transformation and F to the corresponding contragradient transformation. So if we could find "enough" polynomials I_λ in the coefficients of F which are invariant under this action of $SL(N+1)$ then the image of the map given by $C \mapsto (\ldots, I_\lambda(F), \ldots)$ would be $\mathcal{M}_g$.

As of two years ago, this process could be carried out only when char $k = 0$ and C was smooth; and moduli spaces in characteristic p had to be constructed via the much more explicit theory of moduli of abelian varieties (cf. [14] and [15]). Since then, however, two very nice things have been proven:

a) W. Haboush [10] by making a systematic use of Steinberg representations has shown that all reductive groups are geometrically reductive (cf. Remark 1.2. vi). This was independently shown for $SL(n)$, by Processi and Formanek [25], using the idea that the group ring of an infinite permutation group has "radical" zero: i.e. for each $x \in R$, $x \neq 0$, there exists $y \in R$ such that xy is not nilpotent. For a complete treatment of the new situation in characteristic p moduli problems see Seshadri [20].

b) D. Gieseker [9] using the concept of asymptotic stability (cf. 1.17) has established the numerical criterion for stability (c_s of 1.1) for surfaces of general type. Inspired by Gieseker's ideas, the author has extended this method to the "stable" curves of Deligne and Mumford [6]. (These are curves C with dim $H^1(C, \mathcal{O}_C) = g$, ordinary double points but no worse singularities and no smooth rational components meeting the remainder of the curve in fewer than three points; they are important because the most natural compactification $\overline{\mathcal{M}_g}$ of $\mathcal{M}_g$ is the moduli space for stable curves of genus g.) The power of the ideas of Gieseker is by no means exhausted. It looks like nice results may be possible for other surfaces, perhaps even for singular surfaces and the technique suggests several nice problems: in particular, it may lead to a proof of the surjectivity of the period map for K3 surfaces. The new ideas and results of these lectures are largely inspired by Gieseker's results (cf. especially corollary 3.2 below).

My goal is to outline this method and its applications, especially to the completed moduli spaces of curves $\overline{\mathcal{M}_g}$, indicating open problems. The field is moving ahead rapidly and may be greatly simplified in the near future.

We will work in general over an arbitrary ground field k.