§1. Stable points of représentation, examples and Chow forms

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **23 (1977)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **12.07.2024**

§ 1. Stable points of representation, examples and Chow forms

For more details on the notations, definitions and properties which follow see Mumford [14], which we will call G.I.T. or Seshadri [20].

Fix k an algebraically closed field,

G a reductive algebraic group over k (i.e. $G =$ [semi-simple group $\times$ $\mathbf{G}_m^n$]/finite central subgroup),

V an n-dimensional representation of G,

$x \in V$.

There are three possibilities for x whose equivalent formulations are summarized in table 1.1 below.

1.1.

x unstable	x semi-stable	x stable
(a_u) $0 \in \overline{O^G(x)}$	(a_{ss}) $0 \notin \overline{O^G(x)}$	(a_s) i) $O^G(x)$ is closed in V ii) stab (x) is finite
(b_u) $\forall$ non-constant G-invariant homogeneous polynomials f $f(x) = 0$	(b_{ss}) $\exists$ a non-constant G-invariant homogeneous polynomial f s.t. $f(x) \neq 0$	(b_s) i) $\forall y \in V - O^G(x)$, $\exists$ a G-invariant polynomial f s.t. $f(x) \neq f(y)$ ii) tr $\deg_k k(V)^G = \dim V\text{-}\dim G$
(c_u) $\exists$ a 1-PS λ of G s.t. the weights of x with respect to λ are all positive	(c_{ss}) $\forall$ 1-PS's λ of G the weights of x with respect to λ are not all positive	(c_s) for all non-trivial 1-PS's λ of G, x has both positive and negative weights with respect to λ

1.2. REMARKS. i) Recall that a 1-PS (one parameter subgroup) λ of G is just a homomorphism $\lambda\colon \mathbf{G}_m \to G$. Such λ can always be diagonalized in a suitable basis:

$$\lambda(t) = \begin{bmatrix} t^{r_1} & & & 0 \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ 0 & & & t^{r_n} \end{bmatrix}$$

If in this basis $x = (x_1, ..., x_n)$, the set of weights of x with respect to λ is the set of r_i for which $x_i \neq 0$.

ii) Unstable is *not* the opposite of stable, but of semi-stable. We will use non-stable as the opposite of stable.

iii) The important part of stability is the condition: $O^G(x)$ closed in V. In virtually all the cases that will interest us the finiteness of stab (x) will be automatic (but cf. the remark following 1.15).

iv) A point x is stable if it merely has negative weights with respect to every non-trivial 1-PS λ, for then it also has positive weights with respect to λ, namely, its negative weights with respect to λ^{-1}.

v) The proofs of $c_u \Rightarrow a_u \Rightarrow b_u$ and of $b_s \Rightarrow a_s \Rightarrow c_s$ are obvious: for example, if λ is a 1-PS for which all weights of x are positive, then $\lambda(t)\, x \to 0$ at $t \to 0$; i.e. $c_u \Rightarrow a_u$.

vi) The proofs of $a_s \Rightarrow b_s$ and $b_u \Rightarrow a_u$ are achieved by reduction to the special case called geometric reducivity of G. A group G is called geometrically reductive if

a) whenever V_0 is an invariant codimension-1 subspace of a vector space V in which G is represented, there exists an n for which the codimension-1 invariant subspace $V^0 \cdot \mathrm{Symm}^{n-1} V \subset \mathrm{Symm}^n V$ has an invariant 1-dimensional complement.
   But notice that this is the same as saying that

b) whenever $x \neq 0$ is a G-invariant point, then there exists a G-invariant polynomial f such that $f(x) \neq 0$ and $f(0) = 0$. (Just consider x as a functional on the dual $\hat{V}$ and apply a) to its kernel there).

And b) is a special case of $a_s \Rightarrow b_s$. When char $k = 0$ we can take the polynomial f to be linear, for by complete reducibility the invariant subspace generated by x is invariantly complemented. A simple example shows this does not happen in char p. Take $p = 2$, $G = SL(2)$, $V =$ the space of symmetric bilinear functions on k^2, and x a non-degenerate skew-symmetric form ($x \in V$ because $p = 2$!). Then x is $SL(2)$-invariant and there are no G-invariant non-zero linear functionals on V. A quadratic f which does work is the determinant.

vii) The remaining implications $c_s \Rightarrow a_s$ and $a_u \Rightarrow c_u$ are essentially consequences of the surjectivity of the natural map

$$\left\{ \begin{array}{l} \text{1PS's } \lambda \text{ of } G \\ \lambda \colon \mathbf{G}_m \to G \end{array} \right\} \longrightarrow G(k[[t]]) \diagdown G(k((t))) \diagup G(k[[t]])$$

where λ is considered as a $k((t))$-valued point of G by composition with the canonical map

$$\operatorname{Spec} k((t)) \to \operatorname{Spec} k[t, t^{-1}] = \mathbf{G}_m$$

1.3. Let V_{ss} (resp. V_s) denote the Zariski-open cones of semi-stable (resp. stable) points. $V - V_{ss}$ is the Zariski-closed cone of unstable points. The conditions b of 1.1 tell us that if we try to map $\mathbf{P}(V)$ to a projective space by invariant polynomials, we can only hope to achieve a well-defined map on $\mathbf{P}(V)_{ss}$ and an embedding on $\mathbf{P}(V)_s$. From the point of view of quotients this can be expressed by:

PROPOSITION 1.3. *Let* $X = \operatorname{Proj} k[V]^G$. *Then there is a diagram*

$$\begin{array}{ccccc} \mathbf{P}(V) & \supset & \mathbf{P}(V)_{ss} & \supset & \mathbf{P}(V)_s \\ & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_s} \\ & & X & \supset & X_s \end{array}$$

such that i) *if* $x, y \in \mathbf{P}(V)_s$, $\pi_s(x) = \pi_s(y) \Leftrightarrow \exists\, g \in G$ s.t. $x = g\, y$

ii) *if* $x, y \in \mathbf{P}(V)_{ss}$, $\pi(x) = \pi(y) \Leftrightarrow \overline{O^G(x)} \cap \overline{O^G(y)} \cap \mathbf{P}(V)_{ss} \neq \varnothing$.

We now want to look at some examples to illustrate the application of these ideas.

1.4. "BAD" ACTIONS. Using results of T. Kimura and M. Sato [11] [1]), we can give a list of all representations of simple algebraic groups in charac-

[1]) Plus help given by J. Tits.

teristic 0 in which all vectors are unstable. The point is that there are *very* few such representations.

G	V
$SL(W)$	W^i, $\hat{W}^i$, $1 \leqslant k < \dim W$
$SL(W)$, $\dim W$ odd	$\Lambda^2 W$, $\Lambda^2 \hat{W}$ $\Lambda^2 W \oplus \hat{W}$, $\Lambda^2 \hat{W} \oplus W$
$\mathrm{Sp}(W)$	W
Spin (10)	W or $W \oplus W$ where W is a 16-dimensional half-spin representation

1.5. Discriminant. If G is semi-simple and $\operatorname{char} k = 0$ then any irreducible representation V has the form $V = \Gamma(G/B, L)$ for a suitable line bundle L on G/B (B is a Borel subgroup of G). To a point x in V associate the divisor H_x on G/B which is the zero set of the corresponding section. Except in the extremely unusual case that the set of singular H_x is of codimension > 1, there is an irreducible invariant polynomial δ, the discriminant, such that

1) $\delta(x) = 0 \Leftrightarrow H_x$ is singular

2) $V - (\delta = 0)$ consists of semi-stable points.

An interesting case is

Lemma 1.6. *Let $G = SL(n)$, $V = \Lambda^l(k^n)$. If $W \subset k^n$ is a subspace of codimension l then let Φ_W denote the natural map $\Lambda^2 W \otimes \Lambda^{l-2}(k^n) \to \Lambda^l(k^n)$. If $2 < l < n-2$ or n is even $l = 2$ or $n - 2$, then there is a G-invariant δ such that $\delta(x) = 0 \Leftrightarrow x \in \operatorname{Im}(\Phi_W)$ for some W.*

When $l = 2$ and $n = 2m + 1$ we have seen that there are no invariants; corresponding to these cases the Grassmanian of lines in $\mathbf{P}^{2m}$ in its Plücker embedding in projective space has the unusual property that the singular hyperplane sections are of codimension $\geqq 2$ in the set of all such sections.

Question: if not every point of V is unstable, then is the set of singular hyperplane sections H_x of codimension 1 ?

For $l = 2$ and n even or $l = 3$, $n \leqq 8$, one can check that x is unstable $\Leftrightarrow \delta(x) = 0$, hence δ generates the ring of invariants. It would be nice to have a necessary and sufficient condition for a 3-form to be unstable for higher n as well.

1.7. 0-CYCLES. For $G = SL(W)$, $\dim W = 2$,

$$V_n = \mathrm{Symm}^n(\hat{W})$$

= vector space of homogeneous polynomials f of degree n on W,

$\mathbf{P}(V_n)$ = space of 0-cycles of n unordered points on the projective line $\mathbf{P}(W)$, the roots of an f determining the cycle.

If $f = \sum_{i=0}^{n} a_i x^{n-i} y^i$ and λ is the one-parameter subgroup given by $t \mapsto \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$ in these coordinates, then $\lambda(t) f = \sum_{i=0}^{n} a_i t^{n-2i} x^{n-i} y^i$. For f to be stable, the weights $(n-2i)$ associated to the non-zero coefficients of f must lie on both sides of 0: i.e. if $j \geqq n/2$, neither x^j nor y^j divide f.

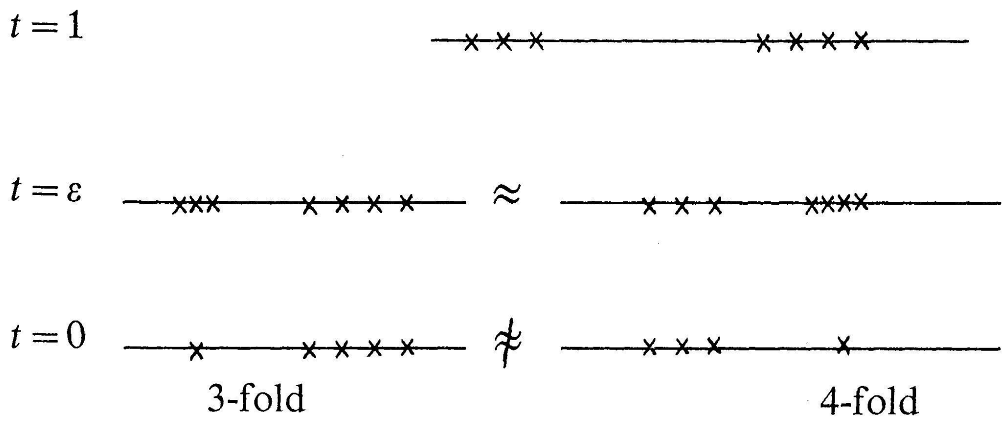
In fact, the stability of f is equivalent to the same condition with respect to all linear forms l: $l^j \nmid f$ if $j \geqq n/2$.

Thus $\mathbf{P}(V_n)_s = \{$0-cycles with no points of multiplicity $\geqq n/2\}$

$\mathbf{P}(V_n)_{ss} = \{$0-cycles with no points of multiplicity $> n/2\}$.

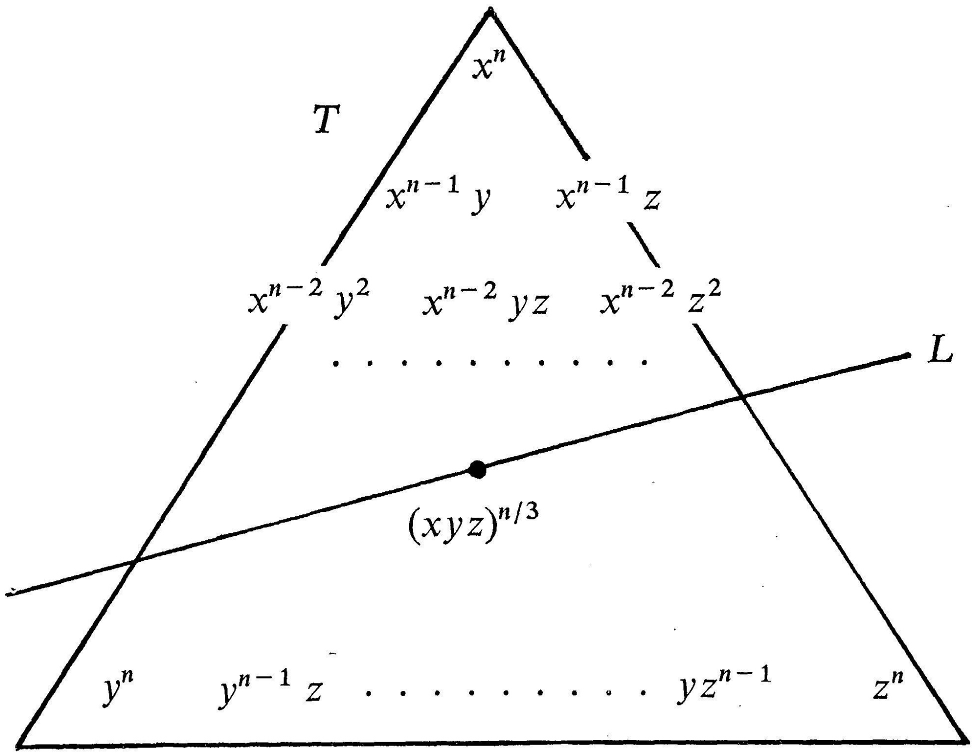
1.8. REMARK. In the example above we can also prove that semi-stability is a purely topological character. I claim that if n is odd and f is unstable then the action of G near $\bar{f} \in \mathbf{P}(V_n)$ is bad: on all open neighbourhoods of the orbit of $\bar{f}$, G acts non-properly and the orbit space is non-Haussdorf. Let's see this for $n = 7$. Consider the following deformations of a 7-point cycle.

(Subscripts indicate multiplicities)



At each intermediate stage the two cycles are projectively equivalent, but the unstable limiting cycle in the right is clearly not equivalent to the limit on the left. In fact, any pair of cycles with the multiplicities indicated on the line $t = 0$ arise in this way as simultaneous limits of projectively equivalent 0-cycles. Moreover, there are cycles of the same type as the left hand limit in any neighbourhood of the orbit of the right limit—just bring a multiplicity one point in towards the triple point; so the orbit space cannot be Hausdorff near the right limit.

1.9. Curves. Here $G = SL(W)$, $\dim W = 3$, $V_n = \mathrm{Symm}^n(\hat{W})$, as before, and a point $f \in V_n$ defines a plane curve of degree n. There is a very simple way to decide the stability of f. Represent f as below by a triangle of coefficients, T.



We can coordinatize this triangle by 3 coordinates i_x, i_y, i_z (the exponents of x, y and z respectively) related by $i_x + i_y + i_z = n$. The condition that a line L with equation $ai_x + bi_y + ci_z = 0$, $(a, b, c) \neq (0, 0, 0)$, should pass through the centre of this triangle is just $a + b + c = 0$; if L also passes through a point with integral coordinates then a, b and c can be chosen integral. It is now easy to check that the weights of the 1-PS

$$t \longmapsto \begin{pmatrix} t^a & & 0 \\ & t^b & \\ 0 & & t^c \end{pmatrix}$$

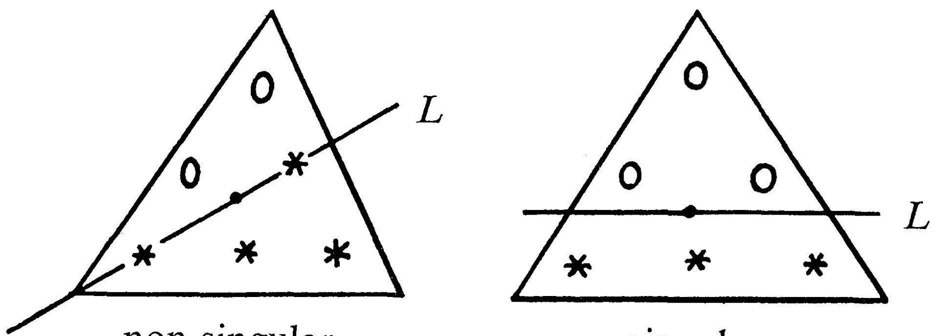
at f are just the values of the form defining L at the non-zero coefficients of f. In suitable coordinates every 1-PS is of this form so:

f is unstable $\Leftrightarrow$ in some coordinates, all non-zero coefficients of f lie to one side of some L

f is stable $\Leftrightarrow$ for all choices of coordinates and all L, f has non-zero (resp. semi-stable) coordinates on both sides of L (resp. f has non-zero coordinates on both sides of L or has non-zero coefficients on L).

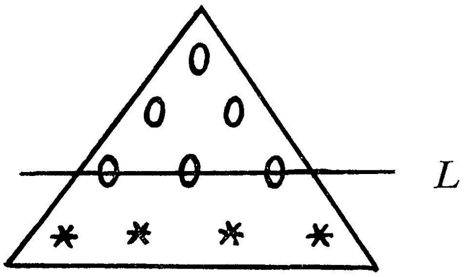
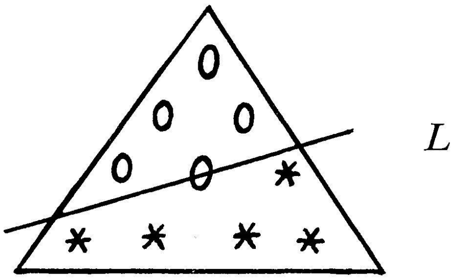
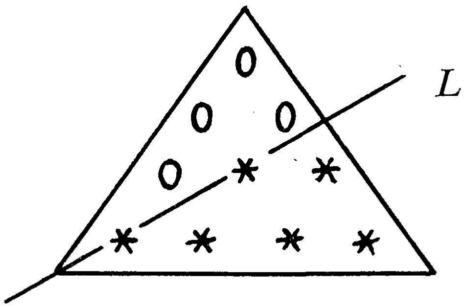
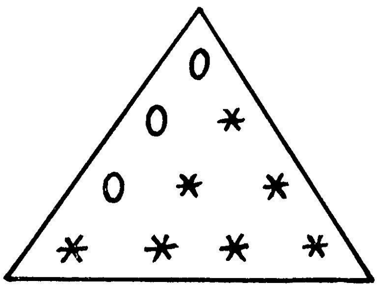
Roughly speaking, a stable f can only have certain restricted singularities. We summarize what happens for small n, showing the "worst" triangle T for f with given singularities, and the associated L when f is not stable.

1.10. $n = 2$: We can achieve the diagram below for a non-singular quadric f by choosing coordinates so that $(1, 0, 0) \in f$ and $z = 0$ is the tangent line there, so f is never stable. We cannot make the xz coefficient of f zero without making f singular so f is always semi-stable; indeed, we know f always has non-zero discriminant. A singular quadric always has a diagram like that on the right: make $(1, 0, 0)$ the double point. Henceforth, we leave the checking of the diagrams to the reader.



non-singular quadric   singular quadric

1.11. $n = 3$: It is well known that in this case the ring of invariants is generated by two invariants, A of degree 4 and B of degree 6. If we set $\Delta = 27A^3 + 4B^2$, then up to a constant the classical j-invariant is just A^3/Δ. The possibilities are:

SINGULARITIES OF f	"WORST" TRIANGLE	STABILITY AND INVARIANTS
f has triple point		unstable $A = B = 0$ j undefined
f has a cusp or two components tangent at a point.		unstable $A = B = 0$ j undefined
f has ordinary double points (this includes the reducible cases: f is a conic and a transversal line, f is a triangle)		semi-stable and not stable $\Delta = 0$ but $A, B \neq 0$ hence $j = \infty$
f smooth		stable $\Delta \neq 0$ j finite

We remark that in this case, we have

$$\begin{array}{ccc} \mathcal{M}_1 & \cong & \mathbf{A}^1 \\ \cap & & \cap \\ \overline{\mathcal{M}_1} & = & \mathbf{P}^2 \end{array}$$

and that the j-invariant is a true modulus. Note that from a moduli point of view all three semi-stable types are equivalent.

1.12. $n = 4$: There are already quite a few diagram types here. Their enumeration can be summarized by saying that f is unstable if and only if f has a triple point or consists of a cubic and an inflectional tangent line; f is stable if and only if f has only ordinary double points or ordinary cusps (i.e. singularities with local equation $y^2 = x^3 +$ higher terms). The remaining f's with a tacnode (a double point with local equation $y^2 = x^4 +$ higher terms) are strictly semi-stable.

1.13. REMARK. The fact that for $n \geqq 4$ curves with sufficiently tame cusps are semi-stable (or even stable!) is a definite problem because

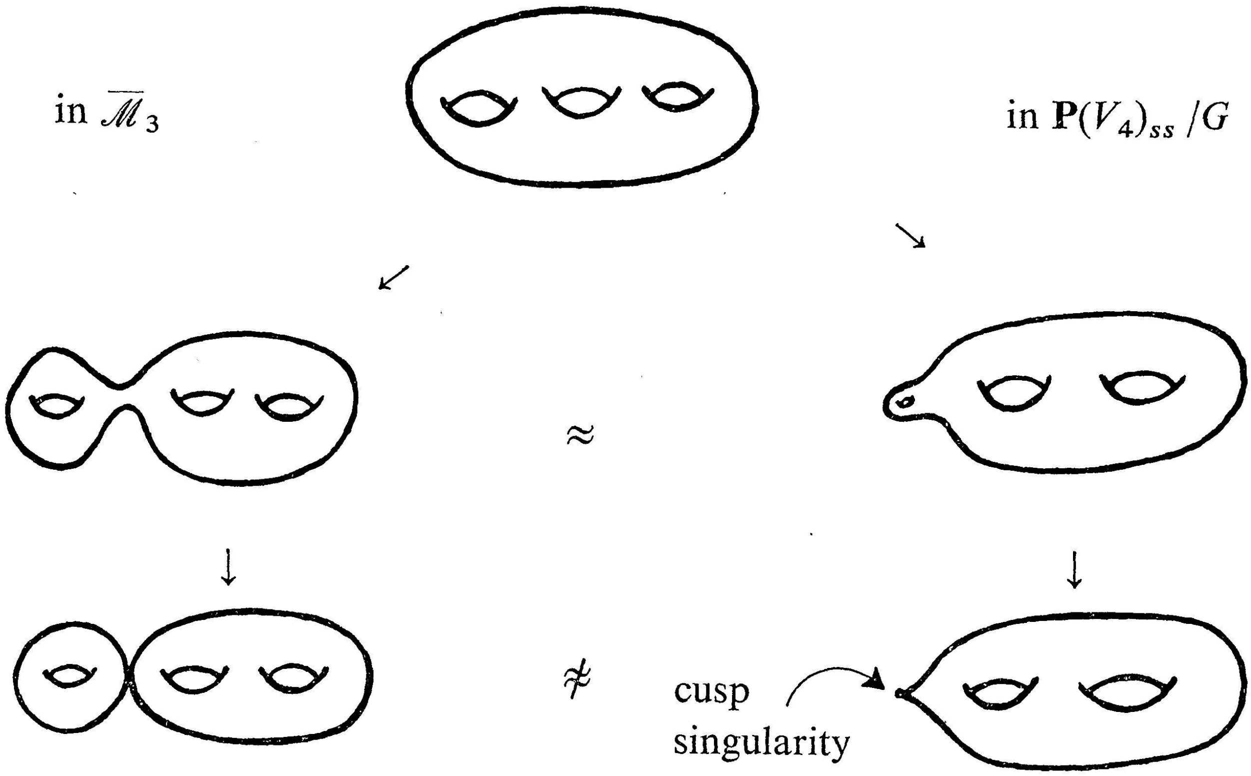
i) such curves do not appear in the good compactification $\overline{\mathcal{M}}_g$ of the moduli space of non-singular curves of genus g. But
ii) if we wish to obtain a compactification of $\mathcal{M}_g$ as the quotient space of some subset of $\mathbf{P}(V_n)$ by G, the natural candidate is $\mathbf{P}(V_n)_{ss}$; so these curves must be let in.

For example, when $n = 4$, we have

$$\begin{array}{ccc} \mathcal{M}_{3,\ \text{non-hyperelliptic}} & \cong & [\mathbf{P}(V_4) - (\delta = 0)]/G \\ \cap & & \cap \\ \mathcal{M}_3 & & \mathbf{P}(V_4)_s/G \\ \cap & & \cap \\ \overline{\mathcal{M}}_3 & \underset{\beta}{\overset{\alpha}{\leftrightarrows}} & \mathbf{P}(V_4)_{ss}/G \end{array}$$

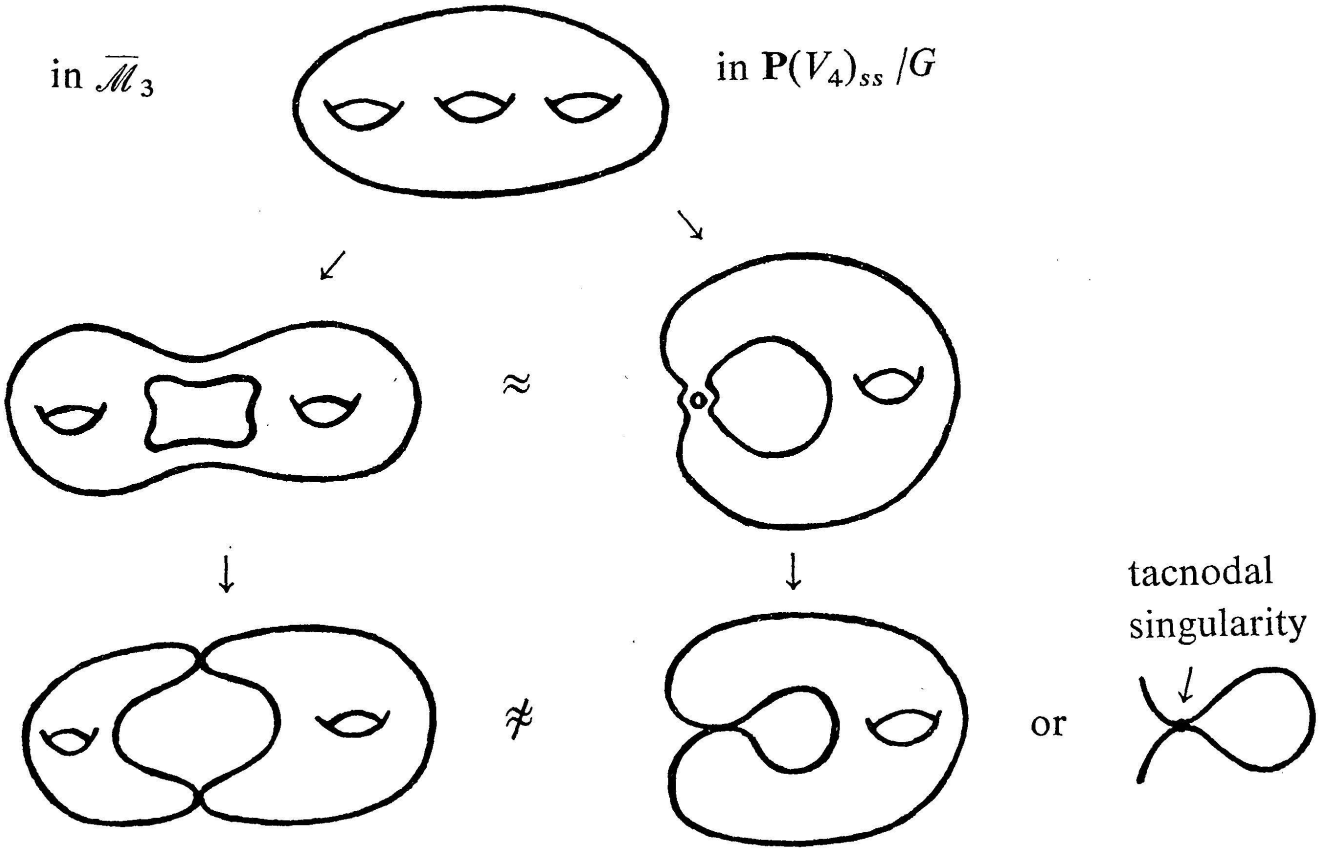
$\overline{\mathcal{M}}_3$ is the moduli space for "stable" curves of genus 3: (see introduction). Recall from Proposition 1.3 that $\mathbf{P}(V_4)_{ss}/G$ is just the projectivization of the full rings of invariants of $\mathbf{P}(V_4)$. The rational maps α and β induced by the top isomorphism enable us to make a topological comparison of these two compactifications. Let's see geometrically how cuspidal curves in $\mathbf{P}(V_4)_{ss}$ prevent α and β from being continuous.

First α: the diagram below shows on the left a deformation on $\mathcal{M}_3$ with limit in $\overline{\mathcal{M}}_3$, and on the right the same deformation followed to its limit in $\mathbf{P}(V_4)_{ss}/G$.



In the limit on the right, the value of the j-invariant of the shrinking elliptic curve has been lost! So α blows up a point representing a curve C with a cusp to the set of points representing joins of an arbitrary elliptic curve with the desingularization $\tilde{C}$ of C. α also blows up the point representing a double conic to the family of all hyperelliptic curves.

As for β, look at the double pinching below:



Here it is the manner in which the tangent spaces of the two branches have been glued at the tacnodal point which has been lost in the limiting curve on the left: this glueing corresponds on the left to the relative rate at which the two pinches are made. Thus β has blown up the point corresponding to the double join of two elliptic curves to a family of tacnodal quartics.

1.14. SURFACES. Here $G = SL(W)$, dim $W = 4$ and $V_n = \mathrm{Symm}^n(\hat{W})$ as before. The technique for determining stability here is essentially that given for curves in 1.9 except that one has a tetrahedron T of coefficients and 1-PS's determine central planes, L: and, of course, the computations required to apply the technique are much more complicated (cf. the case $n = 4$ below). For small n, the situation is summarized below.

n	TYPE OF SINGULARITIES	STABILITY
$n = 2$	non-singular	semi-stable, not stable
	singular	unstable
$n = 3$	non-singular or with ordinary double points of type A1	stable
	ordinary double points of type A2	semi-stable, not stable
	triple points, double curve, higher double points	unstable
$n = 4$ (due to Jayant Shah [26])	singularities at most rational double points, or ordinary double curves possibly with pinch points, but no double *line*, and if reducible then no component a plane, no multiple components	stable
	A triple point whose tangent cone has only ordinary double points; or a double line not as below; or an irrational double point not as below; or a plane plus a cubic meeting in a plane cubic curve with only ordinary double points; or a non-singular quadric counted twice	semi-stable — but not stable
	a) quadruple point, or triple point whose tangent cone has cusp, b) $x = y = 0$ is double line and $f \in (x^2, xyz^2, xy^2, y^3)$ c) a higher double point of form: $f \in (x^2, xy^2, xyz^2, xz^3, y^3z, y^4)$	unstable

1.15. ADJOINT STABILITY.

PROPOSITION 1.15. *Let* G *be any semi-simple group with Lie algebra* $\mathfrak{g}$. *Then* $X \in \mathfrak{g}$ *is unstable* $\Leftrightarrow$ $\operatorname{ad} X$ *is nilpotent.*

Proof: ($\Rightarrow$) From the formula $\operatorname{ad}(\operatorname{Ad} g\,(x)) = \operatorname{Ad} g \circ \operatorname{ad} x \circ \operatorname{Ad} g^{-1}$ it is immediate that the characteristic polynomial $\det(tI - \operatorname{ad} x)$ is G-invariant, hence that is coefficients are invariant functions. If x is unstable, these all vanish so $\operatorname{ad} x$ is nilpotent.
($\Leftarrow$) If $\operatorname{ad} x$ is nilpotent then the $\{ \exp t\,(x) \mid t \in k \}$ is a unipotent subgroup of G which must be contained in the unipotent radical $R_u\,(B)$ of some Borel subgroup B of G. Fix a maximal torus $T \subset B$, so $B = R_u \, . \, T$. Then by the structure theorem of semi-simple groups we can write $\mathfrak{g} = t + \left(\sum_{\alpha > 0} \mathfrak{g}_\alpha \right) + \left(\sum_{\alpha < 0} \mathfrak{g}_\alpha \right)$ where $t = \operatorname{Lie}(T)$ and $\left(\sum_{\alpha > 0} \mathfrak{g}_\alpha \right) = \operatorname{Lie}(R_u\,(B))$. Let χ_α be the character of T, which is associated to $\alpha = (\alpha_i)$ (i.e. if $w \in T$, $y \in \mathfrak{g}_\alpha$ then $\operatorname{Ad}(w)(y) = \chi_\alpha(w)\, y$), and let l be a linear functional on the group of characters of T defining the given ordering: i.e.,

$$l(\chi_\alpha) = \sum_i c_i \alpha_i > 0 \quad \text{if} \quad \alpha > 0 \quad \text{and} \quad l(\chi_\alpha) < 0 \quad \text{if} \quad \alpha < 0 \, .$$

We can always choose l so that all the c_i are integers. If we define a 1-PS $\lambda : \mathbf{G}_m \to T$ by $\lambda(t) = (\ldots, t^{c_i}, \ldots)$, then the weights of X with respect to λ are some subset of $\{ l(\alpha) \mid \alpha > 0 \}$, hence are positive. Thus X is unstable.

REMARK. There are no stable points. One can show that the regular semi-simple elements of $\mathfrak{g}$ have closed orbits of maximal dimension but their stabilizers will be their centralizers, i.e. maximal tori of G, and hence far from finite.

1.16. CHOW FORM. The Chow form is the answer to the problem of describing by an explicit set of numbers a general subvariety $V^r \subset \mathbf{P}^n$. In two cases, the problem has a very easy answer: a hypersurface has its equation F and a linear space L^r has its Plücker coordinates. The Chow form is just a clever combination of these two special cases. Suppose V^r has degree d. There are two ways to proceed

i) If $u = (u_i) \in \mathbf{P}^n$ write H_u for the hyperplance $\sum u_i X_i = 0$. One shows that there is an irreducible polynomial Φ_V such that

$$[V \cap H_u^{(0)} \cap \ldots \cap H_u^{(r)} \neq \varnothing] \Leftrightarrow [\Phi_V(u_i^{(0)}, \ldots, u_i^{(r)}) = 0]$$

Moreover Φ_V is multihomogeneous of degree d in each of the sets of variables $(u_0^{(j)}, \ldots, u_n^{(j)})$, Φ_V is unique up to a scalar, and Φ_V determines V.

ii) If G = Grassmanian of L^{n-r-1}'s in $\mathbf{P}^n$ and $\mathcal{O}_G(1)$ is the ample line bundle on G defined by its Plücker embedding, then the set of $L \in G$ such that $L \cap V \neq \varnothing$ is the divisor D_V of zeroes of some section if $\mathcal{O}_G(d)$ and V and D_V determine each other. (Unfortunately, D_V is almost always a singular divisor.)

These methods give the same result via the identification:

$$\bigoplus_{d=0}^{\infty} \Gamma(G, \mathcal{O}_G(d)) = \left\{ \begin{array}{l} \text{Homogeneous} \\ \text{coordinate} \\ \text{ring of } G \end{array} \right\}$$

$$= \left\{ \begin{array}{l} \text{Subring } W \text{ of } \mathbf{C}[\ldots, U_i^{(j)}, \ldots] \text{ generated} \\ \text{by the Plücker coordinates} \\ P_{i_0, \ldots, i_r} = \det_{(r+1, r+1)} (U_{i_l}^{(j)}),\ i_0 < i_1 < \ldots < i_r \end{array} \right\}$$

Letting W_d be the d^{th} graded piece of W, the identification furnishes an irreducible representation

$$\mathrm{Symm}^d(\Lambda^{r+1}(\mathbf{C}^{n+1})) \twoheadrightarrow W_d \hookrightarrow \bigotimes^{r+1} \mathrm{Symm}^d(\mathbf{C}^{n+1})$$

Thus, although we will usually consider the Chow form as a point of the $SL(n+1)$ representation $\otimes^{r+1}\, \mathrm{Symm}^d(\mathbf{C}^{n+1})$ this form lies in the irreducible piece W_d and can be thought of as defining a divisor on the Grassmanian. For more details on Chow forms, see Samuel [17, Ch. 1 § 9].

1.17. Asymptotic stability. We will say that a variety $V^r \subset \mathbf{P}^n$ is *Chow stable* or simply *stable* if its Chow form is stable for the natural $SL(n+1)$-action. If L is an ample line bundle on V, we say that (V, L) is *asymptotically stable* if

$$\exists\, n_0 \text{ s.t. } \forall n \geqq n_0,\ \Phi_{\Gamma(L^n)}(V) \subset \mathbf{P}^{h^0(L^n)-1} \text{ is stable.}$$

Attention: a stable variety need not be asymptotically stable (nor, of course, vice versa). Indeed, one of the main goals of this exposition is to show that the asymptotically stable curves are exactly the "stable" curves of Deligne and Mumford, and that by using asymptotic stability we can construct $\overline{\mathcal{M}}_g$ as a "quotient" moduli space for these curves.