6. Geometrical characterizations of B*-algebras

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6. Geometrical characterizations of B*-algebras

The first step to a geometrical characterization of B*-algebras among complex Banach algebras was taken in 1956 by Ivan Vidav [52]. To state his result in an appropriate form let us collect some basic ideas and results. For details the reader is referred to the well-written monographs of Frank F. Bonsall and John Duncan [10], [11] on numerical ranges.

Let A be a *unital Banach algebra*, i.e., a Banach algebra with an identity 1 of norm one. A continuous linear functional f on A is called a *state* if $\|f\| = f(1)$. This definition exploits an earlier involution-independent geometrical characterization of the positive functionals on a C*-algebra due to H. Frederic Bohnenblust and Samuel Karlin [9]: a continuous linear functional f on a unital C*-algebra is positive if and only if $\|f\| = f(1)$. Gunter Lumer [34] made strikingly successful use of the generalization of this to define hermitian elements in an arbitrary unital Banach algebra. An element x of a unital Banach algebra A is called *hermitian* if $f(x)$ is real for every state f on A. Clearly, in the special case where A is a C*-algebra, an element $x \in A$ is hermitian if and only if $x^* = x$. Further it turned out that the following conditions for an element x of a unital Banach algebra are equivalent:

1. $f(x)$ is real for every state f on A;
2. $\|1 + i\alpha x\| = 1 + o(\alpha) \quad (\alpha \text{ real});$
3. $\|\exp(i\alpha x)\| = 1 \quad (\alpha \text{ real}).$

In fact, Vidav [52] used the second condition to define hermitian elements in unital Banach algebras. Obviously in the algebra of complex numbers we have

$$|1 + i\alpha x| = 1 + o(\alpha) \quad (\alpha \text{ real})$$

if and only if x is a real number. In the C*-algebra of all bounded operators on a Hilbert space the *self-adjoint operators* (the operators x with $x^* = x$) play the same role as the real numbers in the algebra of complex numbers. Motivated by this observation, Vidav—as he pointed out in a letter to the second named author—asked if the self-adjoint operators could be characterized in a similar way. And, indeed, he was able to show quite easily that an element x in a C*-algebra is self-adjoint if and only if

$$\|1 + i\alpha x\| = 1 + o(\alpha) \quad (\alpha \text{ real}) .$$

Here is his short argument. Let x be any bounded operator on a Hilbert space and write $x = h + ik$, where h and k are self-adjoint. For all real α we have: $\| 1 + i\alpha x \|^2 = \sup \| \xi + i\alpha x\xi \|^2$, where the supremum is taken over all vectors ξ of norm one. We can write:

$$\| \xi + i\alpha x\xi \|^2 = (\xi, \xi) - 2\alpha (k\xi, \xi) + \alpha^2 [\|h\xi\|^2 + \|k\xi\|^2 + i(k\xi, h\xi) - i(h\xi, k\xi)]$$

Hence if $\| \xi \| = 1$, then

$$\| \xi + i\alpha x\xi \|^2 = 1 - 2\alpha (k\xi, \xi) + O(\alpha^2) .$$

Thus $\| 1 + i\alpha x \| = 1 + o(\alpha)$ only if $(k\xi, \xi) = 0$ for every vector ξ. This implies $k = 0$; i.e., x is self-adjoint.

Conversely, if $x = h$ is self-adjoint, then

$$\| \xi + i\alpha h\xi \|^2 = (\xi, \xi) + \alpha^2 \| h\xi \|^2$$

and so $\| 1 + i\alpha h \|^2 = 1 + \alpha^2 \| h \|^2$, which implies $\| 1 + i\alpha h \| = 1 + o(\alpha)$. Thus an element x in a unital C*-algebra is self-adjoint if and only if

$$\| 1 + i\alpha x \| = 1 + o(\alpha) \quad (\alpha \text{ real}) .$$

Further investigations of the set $H(A)$ of hermitian elements in a unital Banach algebra A led Vidav [52] to a rather deep geometrical characterization of B*-algebras.

THEOREM. *Let A be a unital Banach algebra such that:*

i) $A = H(A) + iH(A)$;

ii) *if* $h \in H(A)$ *then* $h^2 = a + ib$ *for some* $a, b \in H(A)$ *with* $ab = ba$.

Then the algebra A has the following properties.

1. *The decomposition $x = h + ik$ with $h, k \in H(A)$ is unique.*

2. *Setting $x^* = h - ik$ if $x = h + ik$ the map $x \to x^*$ is an involution on A. Furthermore for $h \in H(A)$ we have $\| h^2 \| = \| h \|^2$.*

3. *$\| x \|_0 = \| x^*x \|^{1/2}$ defines a B*-norm on A which is equivalent to the original norm.*

Nearly ten years later Barnett W. Glickfeld [24] and Earl Berkson [8] showed independently that A is actually a B*-algebra under its original norm. Their proofs in the commutative case are quite different. Berkson utilized the notion of a semi-inner-product space introduced by Lumer [34] and the theory of scalar type operators as developed by Nelson Dunford [14],

[15], [16]. Glickfeld recognized the importance of the exponential function and obtained the commutative theorem via the hermiticity condition $\| \exp (i\alpha x) \| = 1$ (α real) for $x \in A$. A simplification of his proof was pointed out by Robert B. Burckel [12]. The extension to arbitrary (possibly non-commutative) unital Banach algebras is an immediate consequence of a result of Russo and Dye [44] on unitary operators in C*-algebras (see also Step 6 of the preceding section).

Based on a refinement of the Russo-Dye Theorem, Theodore W. Palmer [41] finally showed that condition ii) in Vidav's theorem is unnecessary and also gave the simplest proof that A is already a B*-algebra under its original norm. Thus the following elegant characterization of B*-algebras was established.

THEOREM. *A unital Banach algebra A admits an involution with respect to which it is a B*-algebra if and only if* $A = H(A) + iH(A)$.

Recently Robert T. Moore [36] gave deep duality characterizations of B*-algebras. He defines *hermitian functionals* on an arbitrary unital Banach algebra A to be those in the real span $H(A')$ of the states on A. It is shown that every functional f in the dual A' of A can be decomposed as $f = h + ik$, where h and k are hermitian functionals. Moore's proof of this uses the usual decomposition of measures. Independently Allan M. Sinclair [51] has given an interesting direct proof in which the measure theory is replaced by convexity and Hahn-Banach separation arguments. Their result is a useful strengthening of the Bohnenblust-Karlin vertex theorem [9] which asserts that the states on a unital Banach algebra separate points in A. Substantial simplifications of the proofs of Moore and Sinclair have been given by L. A. Asimow and A. J. Ellis [4].

Clearly, in the special case where A is a C*-algebra, a continuous linear functional f on A is hermitian if and only if $f(x^*) = \overline{f(x)}$ for all $x \in A$. Moreover, every hermitian functional on a C*-algebra is the difference of two positive functionals (see Corollary 2.6.4 of [13]). We have seen that B*-algebras are characterized among unital Banach algebras as those for which there are *enough* hermitian elements. Moore's duality characterization shows that they may also be characterized as those for which there are *too many* hermitian functionals.

THEOREM. *A unital Banach algebra A admits an involution with respect to which it is a B*-algebra if and only if the dual A' decomposes as a real* *direct sum* $A' = H(A') + iH(A')$; *or, equivalently, iff the hermitian elements in* A *separate points in* A'.

This result reduces an important property of a Banach algebra to properties of its dual space and may play a crucial role in further investigations.

7. Further weakening of the B*-axioms

The result of Russo and Dye on the closed convex hull of the unitaries had an immediate consequence for the further weakening of the B*-axioms. Based on Vidav's theorem [52] or on Glimm-Kadison's proof in [25], as Jacob Feldman [19] observed, the following conclusion results (see [8], [24]).

THEOREM. *A Banach *-algebra* A *with identity is a B*-algebra if and only if* $\| x^*x \| = \| x^* \| \cdot \| x \|$ *whenever* x *and* x^* *commute.*

The assumption of an identity was removed in 1970 by George A. Elliott [17]. A result of Johan F. Aarnes and R. V. Kadison [1] on the existence of an approximate identity in a C*-algebra commuting with a given strictly positive element enabled him to extend the norm on A to A_e so that the algebra A_e still satisfied the B*-condition for normal elements.

In 1972 Vlastimil Pták [42] presented in an excellent forty-five page survey article a simplified treatment of the theory of hermitian Banach *-algebras (that is, Banach *-algebras in which all self-adjoint elements have real spectrum) based on the fundamental spectral inequality

$$| x |_\sigma^2 \leqslant | x^*x |_\sigma .$$

Investigating their connections with C*-algebras, he derived several characterizations of B*-algebras in an elegant way. His article circumvented many difficulties by assuming throughout that the algebras possess an identity element.

In an informal conversation during an ergodic theory conference at Texas Christian University in the summer of 1972 the first named author asked Husihiro Araki if the submultiplicativity condition $\| xy \| \leqslant \| x \| \cdot \| y \|$ was actually necessary in the axioms of a B*-algebra. Some months later Araki and Elliott [2] proved the following two results.