7. Further weakening of the B*-axioms

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direct sum $A' = H(A') + iH(A')$; *or, equivalently, iff the hermitian elements in* A *separate points in* A'.

This result reduces an important property of a Banach algebra to properties of its dual space and may play a crucial role in further investigations.

7. Further weakening of the B*-axioms

The result of Russo and Dye on the closed convex hull of the unitaries had an immediate consequence for the further weakening of the B*-axioms. Based on Vidav's theorem [52] or on Glimm-Kadison's proof in [25], as Jacob Feldman [19] observed, the following conclusion results (see [8], [24]).

Theorem. *A Banach *-algebra* A *with identity is a B*-algebra if and only if* $\| x^*x \| = \| x^* \| \cdot \| x \|$ *whenever* x *and* x^* *commute.*

The assumption of an identity was removed in 1970 by George A. Elliott [17]. A result of Johan F. Aarnes and R. V. Kadison [1] on the existence of an approximate identity in a C*-algebra commuting with a given strictly positive element enabled him to extend the norm on A to A_e so that the algebra A_e still satisfied the B*-condition for normal elements.

In 1972 Vlastimil Pták [42] presented in an excellent forty-five page survey article a simplified treatment of the theory of hermitian Banach *-algebras (that is, Banach *-algebras in which all self-adjoint elements have real spectrum) based on the fundamental spectral inequality

$$| x |_\sigma^2 \leqslant | x^*x |_\sigma .$$

Investigating their connections with C*-algebras, he derived several characterizations of B*-algebras in an elegant way. His article circumvented many difficulties by assuming throughout that the algebras possess an identity element.

In an informal conversation during an ergodic theory conference at Texas Christian University in the summer of 1972 the first named author asked Husihiro Araki if the submultiplicativity condition $\| xy \| \leqslant \| x \| \cdot \| y \|$ was actually necessary in the axioms of a B*-algebra. Some months later Araki and Elliott [2] proved the following two results.

THEOREM 1. *Let A be a *-algebra with a complete linear space norm such that $\| x^*x \| = \| x \|^2$ for all $x \in A$. Then A is a B^*-algebra.*

THEOREM 2. *Let A be a *-algebra with a complete linear space norm such that $\| x^*x \| = \| x^* \| \cdot \| x \|$ for all $x \in A$. Suppose that the involution is continuous. Then A is a B^*-algebra.*

Actually the assertion of Theorem 1 was already implicit under some additional assumptions in a 1961 paper of Vidav [53]. Without knowledge of this interesting paper Araki and Elliott were able to give a rather simple proof of it. On the other hand, Theorem 2 was only established after very long and tedious arguments involving the second dual of a C*-algebra and it would be desirable to have a more elegant proof of it.

Araki and Elliott asked at once if in Theorem 1 or 2 it is enough to assume $\| x^*x \| = \| x \|^2$, respectively $\| x^*x \| = \| x^* \| \cdot \| x \|$, only for all *normal* x (all x with $x^*x = xx^*$). Apparently they were not aware of the following well known counterexample [47]. Let $B(H)$ be the *-algebra of all bounded operators on a Hilbert space H of dimension $\geqslant 2$. The numerical radius of an operator x on H is defined by

$$\| x \|_1 = \sup \{ |(x\xi, \xi)| : \xi \in H, \ \| \xi \| = 1 \}.$$

It is easily seen that $\| \cdot \|_1$ is a complete linear space norm on $B(H)$ with $\frac{1}{2} \| x \| \leqslant \| x \|_1 \leqslant \| x \|$ for all $x \in B(H)$ and $\| x \|_1 = \| x \|$ for all normal $x \in B(H)$, where $\| \cdot \|$ is the usual operator norm (see Chapter 17 of [27]). The norm $\| \cdot \|_1$ has the following properties:

$$\| x^* \|_1 = \| x \|_1 \quad \text{for all} \quad x \in B(H),$$

$$\| x^*x \| \geqslant \| x \|_1^2 = \| x^* \|_1 \| x \|_1 \quad \text{for all} \quad x \in B(H);$$

and

$$\| x^*x \|_1 = \| x \|_1^2 = \| x^* \|_1 \| x \|_1$$

for all normal $x \in B(H)$ but not for all $x \in B(H)$. For another counterexample see the addenda to [2].

In [46] Zoltán Sebestyén proved the following general characterization of B*-algebras.

THEOREM. *Let A be a *-algebra with complete linear space norm such that*

$$\| x^*x \| \leqslant \| x \|^2 \quad \textit{for all} \quad x \in A$$

and

$$\| x^*x \| = \| x \|^2 \quad \textit{for all normal} \quad x \in A .$$

Then A is a B^*-algebra.

In a later paper [48] Sebestyén claimed to prove that continuity of the involution can be dropped from Theorem 2 above. However, G. A. Elliott has pointed out an error in [48]; indeed, on line four of page 212 the series displayed, although convergent, is not shown to converge to the quasi-inverse of $\lambda^{-1}x$. The paper does reduce the problem to the commutative case; but in this case it remains an interesting open question.

8. Applications

Numerous applications of the Gelfand-Naimark theorems appear in the literature. Indeed, utilizing the representation theorem for commutative algebras important theorems in abstract harmonic analysis can be established. For example both the Plancherel theorem and the Pontryagin duality theorem are proved in [33] via the commutative theorem. Further applications to harmonic analysis can be found in [15], [30], [33] and [37]. The representation theorem for commutative algebras can be used to establish important results on compactifications of topological spaces and locally compact abelian groups (see [15], [30] and [33]); it also provides the most elegant method of proof of the spectral theorem for normal operators on a Hilbert space ([15], [30], [33]).

Applications to group representations and von Neumann algebras can be found in [13] and [37]. For applications to numerical ranges of operators see [7], [9], [10], [11] and [34].

In recent years the theory of C*-algebras has entered into the study of statistical mechanics and quantum theory. The basic principle of the algebraic approach is to avoid starting with a specific Hilbert space scheme and rather to emphasize that the primary objects of the theory are the fields (or observables) considered as purely algebraic quantities, together with their linear combinations, products, and limits in an appropriate topology. The representations of these algebraic objects as operators acting on a suitable Hilbert space can then be obtained in a way that depends essentially only on the states of the physical system under investigation. The principal tool needed to build the required Hilbert space and associated representa-