0. Introduction

REPRESENTATION OF COMPLETELY CONVEX FUNCTIONS BY THE EXTREME-POINT METHOD

by Christian Berg

0. Introduction

A function $f:]0, 1[\to \mathbf{R}$ is called completely convex, if it is C^∞ and $(-1)^k f^{(2k)} \geqq 0$ for all $k \geqq 0$. A completely convex function f is called minimal if $f(x) - a \sin(\pi x)$ is not completely convex for any number $a > 0$. Widder showed (cf. [5]) that a completely convex function can be extended to an entire holomorphic function, and in the paper [6] he proved that a minimal completely convex function can be expanded in a Lidstone series. This indicates that the Lidstone polynomials lie on the extreme rays of the cone W of completely convex functions.

The purpose of the present paper is to treat the completely convex functions by the extreme-point method and to obtain the expansion in Lidstone series as a special case of the Choquet representation theorem.

We will proceed as follows: In the topology of point-wise convergence the set W of completely convex functions is a closed, metrizable convex cone. We prove directly that the extreme rays of W are generated by certain polynomials — essentially the Lidstone polynomials — and the function $\sin(\pi x)$. The occurrence of the extreme ray generated by $\sin(\pi x)$ is related to the fact that only minimal completely convex functions can be expanded in Lidstone series.

The cone W has a compact convex base B, and the extreme points of B are determined. It turns out that B is a Bauer simplex, i.e. B is a simplex and the extreme points form a closed set.

The author wants to acknowledge Widder's paper [6] as a source of inspiration. The reason for writing this paper is that we felt it natural to investigate the cone W by the extreme-point method.

Recently Mugler [2] showed that real part of the holomorphic extension of $f \in W$ to the strip $\operatorname{Re} z \in]0, 1[$ is completely convex on each segment $\{x + iy \mid 0 < x < 1\}$. We give a very short proof of this result.