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SOME CONTRIBUTIONS OF BENO ECKMANN TO THE DEVELOPMENT OF TOPOLOGY AND RELATED FIELDS[1]

by Peter HILTON

INTRODUCTION

The work of Beno Eckmann in topology and related fields, extending over a period of 35 years and continuing today just as actively as in those early years, is marked by certain characteristic features which I will attempt to describe in this introduction and to illustrate by selections from his published work in the later sections of my talk.

If one is to look for the distinctive aspects of Eckmann's contributions to mathematics, one might attempt to summarize them as *unification, clarification* and *penetration.* Eckmann has shown quite unique discernment in identifying and developing the relationships between different parts of mathematics; in particular, between algebraic topology on the one hand and linear algebra, homological algebra, group theory and geometry on the other. Simply to say that Eckmann has developed links between algebraic topology and homological algebra is of course to understate the magnitude of his contribution in this area. As a founder of homological algebra, he has helped to forge the fundamental tools of the subject. However, Eckmann's contribution in this direction will be discussed by Saunders MacLane in his talk at this Congress, so that I will do no more than pay tribute to the decisive positive influence that Eckmann has had on the development of this important branch of algebra. I will myself be choosing some examples from Eckmann's publications to illustrate the relationships which he has observed and studied between algebraic topology and the other subjects which I have listed above.

I would like to add that Eckmann has always seen category theory as a means of unification within mathematics. He was one of the earliest contributors to the development of a *point of view* about mathematics which is now so commonplace that young mathematicians find it difficult to believe that it was certainly not obvious to the mathematicians of twenty years ago

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[1]) Presented at the Colloquium on Topology and Algebra, Zurich, April 1977.

who had not come under the influence of Sammy Eilenberg or Saunders MacLane. Moreover, it is possible that those same young mathematicians perusing the literature may not have fully appreciated the significance of Eckmann's role in establishing the point of view to which I refer. Eckmann is not a professional categorist; on the other hand the unqualified benefit of a categorical point of view has been clear to him from his earliest work on group cohomology in 1945, and he has moreover encouraged the development, and the broadening and deepening, of this point of view by inviting to the research institute here in Zürich active exponents of it.

Eckmann achieves clarification primarily by the limpid style of his writing. In both his writing and his lecturing, he follows in the footsteps of his own great teacher Heinz Hopf and himself constitutes a model for his many students. It is difficult, if not impossible, by merely summarizing his work, to demonstrate the clarity of his presentation. There is surely no substitute for reading his own papers. However, it is possible, by taking examples from his published work, to illustrate how very often Eckmann shows what really lies behind an argument or a concept by stripping away much of the superfluous technical detail. It is often the case that an argument in mathematics compels acceptance without really enabling the reader to understand why the statement is true. (The "reader" may even be the author of the argument himself!) Eckmann's own arguments, expressed in his particularly felicitous style of writing, are never of this kind, and, frequently, his papers have been devoted precisely to the clarification of an existing theory and its establishment in an appropriate mathematical context.

It is also characteristic of Eckmann to return to the topics of earlier work in order to demonstrate progress made and the relevance of new tools and techniques to the solution of classical problems. Examples of this significant feature of his work will also be given below.

Of the penetrating nature of Eckmann's work, it is surely unnecessary for me to say a great deal. We would not be gathered here today at this congress to do him honor were it not clear to all of us that his contributions have had a profound effect on the development of our subject. But I would also wish to include under this rubric the penetration of Eckmann's understanding as evidenced by the facility he has to grasp. comprehensively, the significance of new ideas introduced into mathematics.

I find myself in some difficulty in addressing the very congenial task which the organizers of this congress have laid upon me. For, ever since the late 1950's, I have been very closely engaged in joint research with Beno Eckmann. Indeed, over the 15-year period from 1958 to 1973, Eckmann published 37 papers of which 24 were joint papers with myself (and sometimes with a third collaborator, too, principally Urs Stammbach). This long and fruitful collaboration is of course my own adequate testimony to the high regard in which I have always held my good friend Beno Eckmann. But it would perhaps contradict certain canons of good taste if I were to cite our joint work in evidence of the depth of penetration of Eckmann's mathematical insights. Let me therefore only say of that work that I regard my collaboration with Eckmann, and my previous apprenticeship as a student of Henry Whitehead, as the two principal formative elements in my own mathematical growth and maturity. I would only wish to add a reference to the gratification which Eckmann and I felt that a *leit motif* of our joint research, the heuristic duality which we uncovered at the heart of homotopy theory and exploited, received recognition from Norman Steenrod in his listing of principal themes of algebraic topology.

I have said that Eckmann remains as active in mathematical research as ever. This is a source of great delight to us gathered here for this congress, as also for the many mathematicians, all over the world, who derive benefit from his contributions to the progress of our subject. For it is not enough to say that Eckmann remains active; he remains effective, discriminating and entirely contemporary. His most recent work, with Robert Bieri, on Poincaré duality groups and a certain natural generalization of such groups, of which you may hear from Bieri at this congress, exhibits all the qualities to which I have already referred. It is a remarkable tribute to Beno Eckmann that one may say of him that today, at the age of 60, he is still doing his best work.

But, as I feel sure Beno would himself agree, we have had enough of generalities—it is time to get down to some mathematics!

1. Continuous solutions of systems of linear equations

In [8; 1943] Eckmann considered the following problem. Suppose given a system of r linear equations in n unknowns, $r < n$,

$$(1.1) \qquad \sum_{k=1}^{n} a_{ik} x_k = 0, \quad i = 1, 2, \ldots, r < n,$$

where the coefficients a_{ik} are continuous real-valued [1]) functions of a variable u which describes some topological space U, which will usually

[1]) Eckmann had considered the corresponding problem in the complex case in [3; 1942].

be supposed compact metric. A *continuous* solution of the system (1.1) consists of n continuous real-valued functions $x_k(u)$ such that

$$\sum_{k=1}^{n} a_{ik}(u)\, x_k(u) = 0\,;$$

a system of solutions is *linearly independent* if, for each $u \in U$, the solutions are linearly independent in the usual sense; thus a single solution is linearly independent if it vanishes for no value of u. Eckmann supposes the coefficient matrix $(a_{ik}(u))$ to have maximum rank r for all $u \in U$ and asks how many linearly independent solutions the system admits. In answer, he first translates the problem into matrix language: Let $A_{n,r}$ map U to the space of $r \times n$ orthogonal matrices.[1]) Can $A_{n,r}$ be extended, by adjoining rows, to $A_{n,r+l}$? Clearly, if so, and if $A_{n,r}$ is obtained from the coefficient matrix of (1.1) by orthogonalization, then (1.1) admits l linearly independent solutions.

It is, however, Eckmann's next step which we should emphasize. If $m = r + l$, then we consider the fibre-map of Stiefel manifolds $V_{n,m} \xrightarrow{p} V_{n,r}$ with fibre $V_{n-r,m-r}$. Of course, the manifolds $V_{n,m}$ were not called Stiefel manifolds in those days, but Eckmann naturally referred to Stiefel's 1935 paper. Moreover, Eckmann spoke of the *factorization* or *decomposition* ("Zerlegung"),

$$\text{(1.2)} \qquad Z\colon V_{n,m} / V_{n-r,m-r} = V_{n,r}\,.$$

If $f\colon X \to V_{n,m}$ is a map, then f is called the *trace* of $F = Pf\colon X \to V_{n,r}$, with respect to Z, and the problem is to decide whether a given map $A_{n,r}\colon U \to V_{n,r}$ is a trace with respect to Z. Plainly this is a homotopy question, depending only on the homotopy class of $A_{n,r}$; plainly too a constant map is a trace. It follows that if U is contractible then every system (1.1) of maximum rank admits $n - r$ linearly independent solutions—this is a theorem of Wazewski.

Let us now take $U = S^q$. The problem is now one involving *homotopy groups*, and Eckmann comes extraordinarily close to writing down the homotopy sequence of the fibration (1.2); certainly he exploits it effectively, If $q = n - 1$, then we must study homotopy classes of maps $A_{n,m}\colon S^{n-1} \to V_{n,m}$. By projection we get an element of $\pi_{n-1}(V_{n,1})$, that is, an integer c, which we call the *characteristic* of $A_{n,m}$, and it is immediate that a matrix map $A_{n,m}$ exists, with $c = 1$, if and only if there is an $(m-1)$-field on S^{n-1}. Thus if an $(m-1)$-field exists on S^{n-1} every $A_{n,m}$ occurs so that (1.1) has $m - r$ linearly independent solutions.

[1]) We might expect the notation $A_{r,n}$; I have conserved Eckmann's notation.

One may show dy direct matrix arguments that, if $m \geqslant 2$, then $c = 0$ if n is odd (corresponding to the absence of a non-singular vector field on S^{n-1}) and that all even values of c occur if n is even. The question whether odd values of c occur reduces to the question whether $c = 1$ occurs and this in turn leads to the consideration of the Hurwitz-Radon Theorem (see Section 2). Eckmann uses the then existing, scanty knowledge of homotopy groups of Stiefel manifolds to obtain special results (when $q \neq n - 1$)—we would do the same today, but would benefit from our more extensive knowledge.

Indeed, Eckmann himself returned to the question 24 years later when he lectured at a Battelle Rencontre in Seattle [67; 1968]. By this time, of course, Adams had proved his celebrated *Hopf Invariant One Theorem* and the properties of topological K-theory had been substantially developed. Eckmann performed the significant feat of explaining the theory, and its applications—to systems of linear equations, to the existence of (generalized) vector products in $\mathbf{R}^n$, to the parallelizability of spheres, and to the existence of almost-complex structures on spheres—of explaining all this to an audience dominated by theoretical physicists! What testimony to his clarity—and courage!

2. A group theoretical proof of the Hurwitz-Radon Theorem

Immediately following the work discussed above, Eckmann produced [9; 1943] a truly beautiful proof of the celebrated theorem on the composition of quadratic forms. The problem is to determine, given n, those values of p such that there exist n bilinear forms $z_1, \ldots, z_n$ of the variables $x_1, \ldots, x_p$; $y_1, \ldots, y_n$, with complex coefficients, such that the identity

$$(2.1) \qquad (x_1^2 + \ldots + x_p^2)(y_1^2 + \ldots + y_n^2) = z_1^2 + \ldots + z_n^2$$

holds. As formulated by Radon in 1923, the solution is the following. Let $n = u \,.\, 2^{4\alpha+\beta}$ with u odd and $0 \leqslant \beta \leqslant 3$. Then we can find $z_1, \ldots, z_n$ to satisfy (2.1) if and only if $p \leqslant 8\alpha + 2^\beta$. Actually, Radon considered forms with real coefficients, but Eckmann showed explicitly in his proof that a solution of (2.1) for forms with complex coefficients implies a solution for forms with real coefficients. Eckmann's proof is based on the classical theory of (complex) representations of finite groups, together with certain particular results, due to Frobenius and Schur, relating complex to real representations. Before outlining Eckmann's proof, let me quote Eckmann's own remarks justifying his sortie into this well-established field, since it contains a key statement of his approach to what may be called the *systematics* of mathematical exposition. Eckmann wrote:

"Es erscheint aus folgenden Gründen nicht überflüssig, zu den Beweisen von Hurwitz und Radon noch einen dritten hinzuzufügen: einmal ist unser Beweis einfacher und kürzer—dafür operiert er aber mit weniger elementaren Begriffen und Sätzen; ferner sind die Methoden von Hurwitz wie auch von Radon ad hoc konstruiert und liegen außerhalb der sonst in der Algebra üblichen, während wir die Frage in die wohlbekannten Gedankengänge der Darstellungstheorie einordnen, wo sie als schönes Beispiel für die Anwendung allgemeiner Sätze erscheint." [1])

Once again the problem is first replaced by a matrix problem (this step is, of course, common to all three proofs). Thus we seek p complex orthogonal $n \times n$ matrices $A_1, ..., A_p$, such that $A_k A_l' + A_l A_k' = 0$, $l \neq k$. By normalizing we may instead seek $p - 1$ complex orthogonal $n \times n$ matrices $A_1, ..., A_{p-1}$ such that

$$(2.2) \quad A_k^2 = -I, A_k A_l = -A_l A_k, k, l = 1, 2, ..., p-1, k \neq l .$$

Ignoring for the moment the orthogonality condition, Eckmann considers the abstract group G, generated by $(a_1, a_2, ..., a_{p-1}, \varepsilon)$, subject to the relations

$$(2.3) \quad \varepsilon^2 = 1, a_k^2 = \varepsilon, a_k a_l = \varepsilon a_l a_k, k, l = 1, 2, ..., p-1, k \neq l ,$$

and investigates complex representations of G of degree n whereby ε is represented by $-I$. The order of G is 2^p; if $p = 2$, $G = \mathbf{Z}/4$ and the problem is trivial, so we assume $p \geqslant 3$. G has 2^{p-1} non-equivalent representations of degree 1; by counting conjugacy classes it follows that, if p is odd, G has an irreducible representation of degree $f > 1$ and, if p is even, then G has 2 irreducible representations of degrees $f, f' > 1$. It is easy to see (from standard theorems of representation theory) that $f = 2^{\frac{p-1}{2}}$ if p is odd, and that $f = f' = 2^{\frac{p-2}{2}}$ if p is even. Moreover ε will be represented by $-I$ in these irreducible representations, since it cannot be

[1]) "It seems, on the following grounds, not to be superfluous to add a third proof to those of Hurwitz and Radon: on the one hand, our proof is simpler and shorter, although it employs less elementary ideas and theorems; further, the methods of Hurwitz and Radon were constructed in ad hoc fashion and lie outside the domain of standard algebra, whereas we set the problem in the familiar framework of representation theory, *where it serves as a beautiful example for the application of general theorems*". (my italics).

represented by I. Thus the degree n of an arbitrary representation of G of the required kind is given by

(2.4) $$n = m \, . \, 2^{\frac{p-1}{2}}, \; p \text{ odd}; \quad n = m \, . \, 2^{\frac{p-2}{2}}, \; p \text{ even}.$$

It remains to determine which of those representations are equivalent to an orthogonal representation—these will also, according to Frobenius-Schur, be equivalent to orthogonal *real* representations. Corresponding to an irreducible representation D of G, one computes $S = \sum_{g \in G} \chi(g^2)$, where χ is the character function. Then D is real-equivalent if and only if $S > 0$; D is equivalent to its complex conjugate $\bar{D}$ if $S < 0$; and D is not equivalent to $\bar{D}$ if $S = 0$. By a very beautiful application of the elementary theory of complex numbers, Eckmann used this criterion to show that the given irreducible representations of G (whereby ε is represented by $-I$) are real-equivalent (that is, orthogonal-equivalent) if $p \equiv 7, 0, 1 \bmod 8$, and not otherwise. If $p \equiv 3, 4, 5 \bmod 8$ they are equivalent to their complex conjugates; if $p \equiv 2, 6 \bmod 8$ they are not. One may immediately deduce the degrees of real-irreducible real representations of G, and hence show that for a given $n = u \, . \, 2^t$, with u odd, the maximum value of p such that there exists a real (orthogonal) representation of G of degree n, in which ε is represented by $-I$, is given by the rule:

$$\begin{aligned} t &= 4\alpha &&: p = 8\alpha + 1 \\ t &= 4\alpha + 1 &&: p = 8\alpha + 2 \\ t &= 4\alpha + 2 &&: p = 8\alpha + 4 \\ t &= 4\alpha + 3 &&: p = 8\alpha + 8 \, . \end{aligned}$$

This is the Hurwitz-Radon Theorem. Today we know that, when translated into the language of vector fields on spheres, the Hurwitz-Radon number $p - 1$ provides an upper bound on the number of vector fields on S^{n-1} even without the linearity condition; this was, of course, proved by Adams exploiting the techniques of topological K-theory.

3. Complexes with operators

Here perhaps I trespass somewhat on Saunders MacLane's territory. But I do want to exemplify a characteristic feature of Eckmann's thought, whereby he passes freely to and fro between topology and algebra, generalizing both aspects in a constructive and purposeful way. In [33; 1953], which was really the sequel to a pair of short papers [17, 18; 1947] which had appeared some years earlier in the Proceedings of the National Academy of Sciences, Eckmann considered a generalization of the algebraic constructions involved in studying the homology of covering spaces.

Let R be a unitary ring and let C be an R-complex (that is, a chain complex such that each C_p is an R-module and each boundary $\partial\colon C_p \to C_{p-1}$ an R-homomorphism). Let J be an abelian group and let $\Phi = \mathrm{Hom}\,(R, J)$ be the group of additive homomorphisms of R into J, turned into an R-module by the rule

$$(s\varphi)(r) = \varphi(rs), r, s \in R, \varphi\colon R \to J \tag{3.1}$$

If Ψ is a submodule of Φ we say that the p-cochain f of C, with values in J, is of type Ψ if, for each $c_p \in C_p, f(rc_p)$, as a function of $r \in R$, belongs to Ψ. It is easy to check that then the coboundary δf is again of type Ψ, so that we may define the Ψ-cohomology groups of C with coefficients in J, written $H^p_\Psi(C, J)$. Among the examples which Eckmann gives of Ψ-cohomology are the following:

(a) If $\Psi = \Phi$, we simply get the cohomology groups $\check{H}^p(C, J)$ of C, regarded as a complex of abelian groups, with values in J.

(b) If J is an R-module and Ψ consists of the R-homomorphisms from R to J, then a cochain of type Ψ is an *equivariant* cochain and $H^p_\Psi(C, J)$ is just the equivariant cohomology group which we will write simply as $H^p(C, J)$. Clearly we have here an isomorphism $\Psi \cong J$ given by $\varphi \mapsto \varphi(1)$. This isomorphism suggests the general conclusion embodied in the isomorphism (3.2) below.

(c) If Q is a subring of R and J is a Q-module, we may take Ψ to consist of all Q-homomorphisms $R \to J$. If C_Q denotes the complex C regarded as a Q-complex, then $H^p_\Psi(C, J) = H^p(C_Q, J)$. Plainly, (c) generalizes (a) and (b).

(d′) Let A be a group, B be a subgroup of A. If we take $R = \mathbf{Z}[A]$, $Q = \mathbf{Z}[B]$ in (c) we obtain the group Ψ of functions ψ from A to the B-module J such that $\psi(ba) = b\psi(a)$. The A-module structure on Ψ is said to be *induced* by the B-module structure on J (it corresponds very precisely to the *induced representation* of A, induced by the representation of B by J).

(d″) If A is a group and J an abelian group, and if $R = \mathbf{Z}[A]$ we may take Ψ to consist of those functions $\Psi\colon A \to J$ which vanish almost every-where on A. The resulting cohomology group $H^p_\Psi(C, J)$ is called *A-finite* and denoted $H^p_{A-\mathrm{fin}}(C, J)$. If $J = \mathbf{Z}$, $\Psi \cong \mathbf{Z}[A]$; an isomorphism (of A-modules) is given by $\psi \to \Sigma\ \psi(a)\, a^{-1}$.

Eckmann unifies all those examples, coalescing them into example (b), by means of the isomorphism

$$H^p(C, \Psi) \cong H^p_\Psi(C, J)\,, \tag{3.2}$$

induced, at the cochain level, by $f \mapsto g$, where

(3.3) $g(c) = f(c)(1)$, $c \in C_p$, $f\colon C_p \to \Psi$ (equivariant), $g\colon C_p \to J$;

he then applies (3.2) in various contexts. The fact that the isomorphism (3.2) is now a commonplace certainly does not detract from its significance—on the contrary!

Among the applications, let us mention the following. Les S be a nice topological space, so that we can construct covering spaces of S. Let B be a subgroup of the fundamental group A of S, let S_B be the covering space of S corresponding to B, let J be a B-module and let Ψ be the induced A-module in the sense of (d'). We then have an isomorphism of singular cohomology with local coefficients,

$$H^p(S_B, J) \cong H^p(S, \Psi)\,. \tag{3.4}$$

A further, very significant application made by Eckmann in [33] and further developed in [35; 1953] is to the (generalized) *transfer*. We will not go into that here, but instead will turn to the theory of *ends* of groups. If A is a finitely presented group and P a compact polyhedron with $\pi_1 P = A$, then the ends of A may, following Hopf, be defined in terms of the universal cover $\tilde{P}$ of P; since they refer to the "infinite components" of $\tilde{P}$, the theory of ends is only of interest if A is not finite. If C is the chain-complex of some simplicial decomposition of $\tilde{P}$, then Specker proved that $H^1(C, \mathbf{Z}[A])$ is a free abelian group whose rank is $e - 1$, where e is the number of ends [1]) of A. Indeed, one has

$$H^1(C, \mathbf{Z}[A]) \cong H^1(A, \mathbf{Z}[A]) \cong D/D_0\,, \tag{3.5}$$

where D is the group of (Fox) *derivations* from A to $\mathbf{Z}[A]$ and D_0 is the subgroup of inner derivations. Now according to (d″) and (3.2), $H^1(C, \mathbf{Z}[A])$

[1]) Hopf proved that, if A is not finite, then e can only take the values $1, 2, \infty$.

$\cong H^1_{A-\mathrm{fin}}(C, \mathbf{Z})$. But since the chain complex C/A, obtained from C by factoring out the operations of A, is of finite type, it follows that

$$H^1_{A-\mathrm{fin}}(C, \mathbf{Z}) = H^1_{\mathrm{fin}}(C, \mathbf{Z}), \tag{3.6}$$

the first cohomology group of $\tilde{P}$ with integer coefficients *based on finite cochains*, that is, on cochains which vanish on almost every simplex of $\tilde{P}$.

4. Spaces with means

In his talk at the 1950 International Congress of Mathematicians [32], the first international congress to be held after the second world war, Eckmann addressed himself to the question of the existence on a topological space X of a map $\mu: X^n \to X$, where X^n is the n-$^{\text{th}}$ cartesian power of X, which should be symmetric in the n variables and should satisfy $\mu(x, x, ..., x) = x$, $x \in X$. He returned to the theme in the paper he presented on the occasion of the celebration of the sixtieth birthday of Heinz Hopf [39; 1954], and it is therefore appropriate that I should refer to it here.

The methods used by Eckmann to study this problem were, of course, those of homotopy theory; they are thus very different from those of Aumann who first considered the problem in 1943. On the other hand, they do enable one to investigate the more general concept of *homotopy-mean*, whereby we understand that the map μ is only required to satisfy the conditions imposed above *up to homotopy*. This approach was explicitly followed in the sequel [55; 1962] where, in collaboration with T. Ganea and the present writer, Eckmann effectively gave a complete solution of the problem, or, as one may say, killed it!

In [39], Eckmann showed that if X admits an n-mean, so do its homotopy groups and homology groups. So far as the homotopy groups are concerned this is a "trivial" consequence, in the sense that it follows on categorical grounds from the fact that the homotopy group functor is *product-preserving*; however, the argument relating to the (integer-valued) homology groups was a rather subtle application of the Künneth Theorem. Moreover, an n-mean, $n \geqslant 2$, can only exist in a group if the group is abelian, and then it exists (and is unique) if and only if the group admits unique division by n. Indeed the n-mean $\mu: G^n \to G$ is simply

$$\mu(x_1, x_2, ..., x_n) = \frac{1}{n}(x_1 + x_2 + ... + x_n). \tag{4.1}$$

Eckmann used this criterion in establishing that the homology groups of X admitted an n-mean if X admitted an n-mean; and to show that if the compact polyhedron X admits a (homotopy) n-mean for all n, then X is contractible. He raised many questions, among them whether the existence in such a space X of an n-mean for some $n \geqslant 2$ might imply the contractibility of X. This question was answered positively in [55].

In that paper, the idea of an n-mean was first placed in its appropriate categorical setting, so that the trivial (= categorical) aspects of the theory of n-means could first be exhibited. In particular this permitted the description of the dual concept of an *n-comean.* However, insofar as groups are concerned, the situation for the existence of n-comeans must be distinguished from that for the existence of n-means. Fix $n \geqslant 2$. Then in the category of groups, only the trivial group admits an n-comean; in the category of abelian groups, A admits an n-comean if and only if it admits an n-mean. On the other hand, there are many compact polyhedra admitting an n-comean; if m is prime to n and $k \geqslant 2$, and if X is the Moore space having $\mathbf{Z}/m$ as its single non-vanishing homology group in dimension k, then X admits an n-comean. On the other hand, the dual of Eckmann's result in [39] holds: if the compact polyhedron X admits an n-comean for all n, then X is contractible.

Reverting to Eckmann's question, one shows first that if X is a compact polyhedron admitting an n-mean, $n \geqslant 2$, then X is an H-space; that is, X admits a continuous multiplication with two-sided unity element. However, Browder showed that such spaces satisfy Poincaré duality. Thus, were X non-contractible, its top-dimensional (non-vanishing) homology group $H_k X$ would be cyclic infinite, and therefore could not admit division by n, contradicting Eckmann's result in [39]. Thus the application of Browder's deep theorem kills the homotopy-theoretic interest of means in compact polyhedra. It is pleasant to record that the obituary article [55] was published in *Studies in Mathematical Analysis and Related Topics*, symbolic testimony to the value of Eckmann's constant search for relations between the different branches of mathematics. But perhaps, like a famous obituary of Bertrand Russell, the notice of the death of homotopy n-means was premature. For localization theory, a powerful new tool in homotopy theory, has rekindled interest in non-compact polyhedra, and there are surely interesting examples of such polyhedra admitting n-means—for example, Eilenberg-MacLane spaces for $\mathbf{Z}\left[\frac{1}{n}\right]$-local spaces.

5. Simple homotopy type

Our final example of Eckmann's work brings us very much into the modern era. It will be understood why I have eschewed the temptation to deal with his very extensive contributions between 1958 and 1973 in detail. However, in 1970, he published a paper with Serge Maumary [70], dedicated to Georges de Rham, to which it will surely repay us to give some attention.

In 1950 Henry Whitehead recast in algebraic terms the theory of simple homotopy types which he had introduced many years earlier. In this theory one considers the collection of finite simplicial complexes [1]) homotopically equivalent to a given complex X and introduces the finer classification of *simple* equivalence into this collection. Whitehead showed that a single invariant sufficed to classify homotopy equivalences modulo simple equivalences; this invariant is now known as *Whitehead torsion* and is an element of an abelian group constructed functorially out of $\pi_1 X$. The importance of this theory has come to be recognized more clearly in recent years with the rise of differential topology and algebraic K-theory; indeed, the Whitehead group $\mathrm{Wh}\,\pi$, to which the Whitehead torsion belongs ($\pi = \pi_1 X$) is a quotient of $K_1\,\pi$. Chapman has proved the topological invariance of Whitehead torsion using the techniques of infinite-dimensional topology.

In [70] Eckmann and Maumary give a purely geometric description of the Whitehead group $\mathrm{Wh}\,X$. They start with the classical description of a simple equivalence $s\colon Y \to Y'$ between finite cell complexes, as a sequence of elementary expansions and contractions. For a given finite cell complex X, they then introduce an equivalence relation into the family of maps emanating from X with target a finite cell complex. Thus $f\colon X \to Y$ and $f'\colon X \to Y'$ are equivalent if there exists a simple equivalence $s\colon Y \to Y'$ such that $sf \simeq f'$. Let $A(X)$ be the set of equivalence classes thus defined. One may introduce a binary operation into $A(X)$ by means of the *homotopy push-out*—here we adopt a description equivalent to but not identical with that of [70]. Thus let $g\colon X \to Y$, $h\colon X \to Z$ be maps which we may assume cellular. Replace one or both of g, h by cofibrations; this may be done by the mapping cylinder construction. Thus, assume that g, h are cofibrations and construct the topological push-out of g, h—this will be the double mapping cylinder if g, h are embeddings in mapping cylinders:

[1]) There is no problem in replacing simplicial complexes by (finite) CW-complexes.

(5.1)
$$\begin{array}{ccc} X & \xrightarrow{g} & Y \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle u} \\ Z & \xrightarrow[v]{} & M \end{array}$$

Then if $< g >$ stands for the class of g, etc., we define, from (5.1),

(5.2)
$$< g > + < h > \; = \; < ug >$$

Next one makes A a functor to sets, also by means of the homotopy push-out: if $f \colon X \to X'$ we take a cofibration g in the class $< g >$ and define

(5.3)
$$f_* < g > \; = \; < \bar{g} >$$

from the push-out

(5.4)
$$\begin{array}{ccc} X & \xrightarrow{g} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \bar{f}} \\ X' & \xrightarrow[\bar{g}]{} & Y' \end{array}$$

The fact that f_* is a homomorphism with respect to the addition (5.2) is proved by purely categorical arguments; so too is the fact that $A(X)$ is an abelian monoid under the addition (5.2); of course the zero of $A(X)$ is $< 1_X >$.

Now let $E(X)$ be the subset of $A(X)$ consisting of classes $< g >$ such that g is a homotopy equivalence. It is plain that $E(X)$ is a submonoid; however, it is actually an abelian *group*. First, E is a functor, that is, $f_* EX \subseteq EX'$ if $f \colon X \to X'$. Second, let $g \colon X \to Y$ be a homotopy equivalence with homotopy inverse $g' \colon Y \to X$ and set $< h > \; = g'_* < g' >$, $h \colon X \to Z$, so that $< h > \in EX$. Then from (5.1)-(5.4) it follows that

$$< g > + < h > \; = \; < g' g > \; = \; < 1_X > \; = 0 \, .$$

We have thus defined a functor $E \colon P \to Ab$, where P is the category of finite cell-complexes and Ab is the category of abelian groups. Finally, Eckmann and Maumary prove that EX depends only on the 2-skeleton of X; a general argument then enables them to deduce that E factors through the fundamental group functor.

The gain in conceptual simplicity achieved by this geometric viewpoint is substantial; of course, the hard calculations remain to be done to compute the Whitehead group. One may compare the achievement of this paper with that of [34; 1953], in which Eckmann and Schopf produce a very significant simplification and clarification of the concept of injective hull of a module and a very easy, natural proof of its existence (first proved by Reinhold Baer); or with a very recent paper [81; 1976], in which Eckmann gave a remarkably simple proof of the Dyer-Vasquez theorem that the complement of a higher-dimensional knot $S^{n-2} \subseteq S^n$, $n \geqslant 4$, is never aspherical unless the knot group is infinite cyclic (thus, if $n \geqslant 5$, unless the knot is unknotted).

The story goes on. I have on my desk the latest manuscript, a joint paper by Eckmann and Bieri, completed in the spring of 1977, entitled "Relative Homology and Poincaré duality for group pairs". As I have said, Beno Eckmann remains active and effective—but more is true. The Eckmann touch remains as sure as ever!

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