# 4. Ex-HOMOTOPY THEORY

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$$(t_1 a_1', \ldots, t_n a_n') \qquad (a_i' \in A_i', t_i \in I)$$

where $t_1^2 + \ldots + t_n^2 = 1$. The radius vector through $(t_1, \ldots, t_n)$ meets the boundary of the $n$-cube $I^n$ in a point $(x_1, \ldots, x_n)$, say, where at least one coordinate is equal to 1. Thus a pointed $G$-homotopy equivalence

$$l \colon \Sigma(A_1' * \ldots * A_n') \to \Sigma A_1' \wedge \ldots \wedge \Sigma A_n'$$

is given by

$$l\left((t_1 a_1', \ldots, t_n a_n'), s\right) = \left((s x_1, a_1'), \ldots, (s x_n, a_n')\right).$$

Clearly $l$ is equivariant with respect to the action of the symmetric group on the suspension of the multiple join and on the multiple smash product.

In particular, take $G = O(m)$ and $A_i' = S^{m-1}$, for all $i$. Let $u$ be a permutation of the multiple join and $v$ the corresponding permutation of the multiple smash product. We distinguish cases according as to whether the degree of the permutation is even or odd. In the even case $u$ is $G$-homotopic to the identity $1_n$ on the $n$-fold join, using elementary rotations as before, and hence $v$ is pointed $G$-homotopic to the identity $1_n$ on the $n$-fold smash product. In the odd case it follows similarly that $u$ is $G$-homotopic to $1_{n-1} * a$, hence $v$ is pointed $G$-homotopic to $1_{n-1} \wedge \hat{a}$. Taking $n = 3$, therefore, we see that the automorphisms which appear in (2.2) are trivial, in this example, and so

$$3\left[\Sigma_* \iota_m, [\Sigma_* \iota_m, \Sigma_* \iota_m]\right] = 0 \tag{3.3}$$

in $\pi_G(S^{3m+1}, S^{m+1})$, where $\iota_m$ denotes the pointed $O(m)$-homotopy class of the identity on $S^m$. It is easy to see, incidentally, that the Whitehead square $[\Sigma_* \iota_m, \Sigma_* \iota_m] \in \pi_G(S^{2m+1}, S^{m+1})$ is of infinite order, for all $m \geqslant 2$.

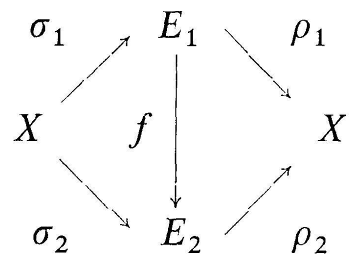
## 4. Ex-homotopy theory

For our second example of an alternative homotopy theory we take the category of ex-spaces (see [7] for details), which is an enlargement of the category of sectioned bundles mentioned earlier. We recall that, with regard to a given space $X$, an *ex-space* consists of a space $E$ together with maps

$$X \xrightarrow{\sigma} E \xrightarrow{\rho} X$$

such that $\rho\sigma = 1$. We refer to $\rho$ as the *projection*, to $\sigma$ as the *section*, and to $(\rho, \sigma)$ as the *ex-structure*. Let $E_i$ $(i = 1, 2)$ be an ex-space with ex-structure

$(\rho_i, \sigma_i)$. We describe a map $f : E_1 \to E_2$ as an *ex-map* if $f\sigma_1 = \sigma_2, \rho_2 f = \rho_1$, as shown in the following diagram.




In particular we refer to $c = \sigma_2 \rho_1$ as the *trivial ex-map*. We also describe a homotopy $h_t : E_1 \to E_2$ as an *ex-homotopy* if $h_t$ is an ex-map throughout. The set of ex-homotopy classes of ex-maps is denoted by $\pi_X(E_1, E_2)$ and the class of the trivial ex-map by 0.

In particular, suppose that $E_i$ is a sectioned bundle with locally compact fibre. For each point $x \in X$ the fibre $\rho_i^{-1}(x)$ is equipped with basepoint $\sigma_i(x)$. Consider the fibre bundle $M = M_X(E_1, E_2)$ which is formed, in the usual way (see [2]) from the function-spaces of pointed maps $\rho_1^{-1}(x) \to \rho_2^{-1}(x)$. To each ex-map $f : E_1 \to E_2$ there corresponds a cross-section $f' : X \to M$, where $f'(x)$ is given by the restriction of $f$ to the fibre over $x$, and conversely every such cross-section determines an ex-map. We shall exploit this correspondence in the next section.

Now let $P$ be a principal $G$-bundle over $X$, where $G$ is a topological group. For any pointed $G$-space $A$ the pointed $G$-bundle $P_\# A$ can be regarded as an ex-space, and similarly with pointed $G$-maps. Thus $P_\#$ constitutes a functor from the category of pointed $G$-spaces to the category of ex-spaces, and determines a function

$$P_\# : \pi_G(A_1, A_2) \to \pi_X(E_1, E_2),$$

where $A_i$ $(i = 1, 2)$ is a pointed $G$-space and $E_i = P_\# A_i$. Of course, in general $P_\#$ is neither injective nor surjective.

As we have seen in § 1 a functor $F$ in the category of pointed $G$-spaces defines a functor $F$ in the category of sectioned $G$-bundles; in many cases such a functor can be extended to the category of ex-spaces. For example, the suspension functor $\Sigma$ and the loop-space functor $\Omega$ can be so extended, also the binary functors product $\times$, wedge $\vee$, and smash $\wedge$. Similarly the notions of Hopf ex-space, etc.; can be introduced, following the standard formal procedure, so that $P_\#$ transforms Hopf $G$-spaces into Hopf ex-spaces, and so forth. Note that $\Sigma E$ is cogroup-like and $\Omega E$ group-like, for any ex-space $E$.

The Whitehead product theory for ex-spaces has been worked out by Eggar [4]. His definition is such that if $A$, $B$, $Y$ are as in §2 and $\alpha \in \pi_G(\Sigma A, Y)$, $\beta \in \pi_G(\Sigma B, Y)$ then

$$(4.1) \qquad [P_{\#}\alpha, P_{\#}\beta] = P_{\#}[\alpha, \beta]$$

in $\pi_X(\Sigma(P_{\#}A \wedge P_{\#}B), P_{\#}Y)$. Since we shall only be concerned with elements in the image of $P_{\#}$ we can introduce (4.1) as a piece of notation, without going into the details of Eggar's theory.

## 5. The register theorem

In this section we suppose that $X$ is a finite simply-connected $CW$-complex, although the results obtained can no doubt be generalized. We define the *register* $\operatorname{reg}(X)$ of $X$ to be the number of positive integers $r$ such that, for some abelian group $A$, the cohomology group $H^r(X; A)$ is non-trivial. If $X$ is a sphere, for example, then $\operatorname{reg}(X) = 1$.

Let $p: M \to X$ be a fibration with fibre $N$. If a cross-section $s: X \to M$ exists then $sp: M \to M$ is a fibre-preserving map which is constant on the fibre. Conversely if $k: M \to M$ is a fibre-preserving map which is nulhomotopic on the fibre then $M$ admits a cross-section as shown by Noakes [11]. We use similar arguments to prove

Theorem (5.1). *Let $k: M \to M$ be a fibre-preserving map such that $l: N \to N$ is nulhomotopic, where $l = k|N$, and let $s, t: X \to M$ be cross-sections. Then $k^r s$ and $k^r t$ are vertically homotopic, where $r = \operatorname{reg}(X)$.*

The $n$-section ($n=0, 1, \ldots$) of the complex $X$ is denoted by $X^n$. Since $X$ is connected we have a vertical homotopy of $s$ into $t$ over $X^0$. This starts an induction. Suppose that for some $n \geqslant 1$ and some $q = q(n) \geqslant 1$ we have a vertical homotopy of $k^q s$ into $k^q t$ over $X^{n-1}$, so that the separation class

$$d = d(k^q s, k^q t) \in H^n(X; \pi_n(N))$$

is defined. If the cohomology group vanishes then $d = 0$ and $k^q s \simeq k^q t$ over $X^n$. But in any case the induced endomorphism $l_*$ of $\pi_n(N)$ is trivial, by hypothesis, and so $d$ lies in the kernel of the coefficient endomorphism $l_{\#}$ determined by $l_*$. Therefore

$$d(k^{q+1}s, k^{q+1}t) = l_{\#}d = 0,$$