SINGULAR INTEGRAL EQUATION CONNECTED WITH QUASICONFORMAL MAPPINGS IN SPACE

A SINGULAR INTEGRAL EQUATION CONNECTED WITH QUASICONFORMAL MAPPINGS IN SPACE

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Dedicated to Albert Pfluger for his seventieth birthday

1. INTRODUCTION

This paper continues the author's investigation of two differential operators, S and S^*, which arise naturally in the study of infinitesimal quasiconformal mappings in n dimensions (see References). If Ω is open in $\mathbf{R}^n$ the operator S acts on functions $f : \Omega \to \mathbf{R}^n$ and has values $Sf \in SM_n$ where SM_n is the space of symmetric $n \times n$ matrices with zero trace. Definitions are in Sec. 2.

A key question is the solvability of the inhomogeneous equation $Sf = v$. For $n = 2$, Sf can be identified with the complex derivative $f_{\bar{z}}$ of a complex-valued function, and the problem is that of recovering f from $f_{\bar{z}}$. As well known, this problem has always a solution, and it is given by the generalized Cauchy formula, also known as Pompeiu's formula. For $n > 2$ the right hand member v, an SM_n-valued function, must satisfy certain conditions, which are known in principle, as limiting cases of the Weyl-Schouten conditions of vanishing conformal curvature.

These conditions, although explicit, are quite intractable. It is therefore rather surprising that a necessary and sufficient condition for $Sf = v$ to be solvable can be expressed as a singular homogeneous integral equation satisfied by v. This integral equation can be treated by the methods of Calderon and Zygmund.

2. DEFINITIONS AND NOTATIONS

A quasiconformal homeomorphism $F : \Omega \to F(\Omega)$ is known to be differentiable almost everywhere. We denote its Jacobian matrix by DF. The normalized Jacobian is $XF = (\det DF)^{-1/n} DF$, and $MF = {}^tXF \cdot XF$

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is the normalized and symmetrized Jacobian; it carries the quasiconformal data of the mapping.

The Riemannian metric $ds^2 = {}^t dx\,(MF)\,dx$ is conformally flat, a condition expressed by the vanishing of the conformal curvature tensor. For $n = 3$ this tensor is identically zero, but there is instead an integrability condition.

Let $F(x, t)$ be a one-parameter family of homeomorphisms such that $F(x, 0) = x$, $\dot{F}(x, 0) = f(x)$. Under suitable regularity conditions $(DF)^{\boldsymbol{\cdot}}_0 = Df$, $(XF)^{\boldsymbol{\cdot}}_0 = Df - \frac{1}{n}\, tr\, Df \cdot 1_n$, and $(MF)^{\boldsymbol{\cdot}}_0 = Df + {}^t Df - \frac{2}{n}\, tr\, Df \cdot 1_n$. This motivates introducing the differential operator S defined by

$$(Sf)_{ij} = \frac{1}{2}(D_i f_j + D_j f_i) - \frac{1}{n}\,\delta_{ij}\, D_k f_k\ .$$

(The summation convention is in force in this paper). Note that Sf has values in SM_n.

There is a formal adjoint S^* which maps SM_n-valued functions on $\mathbf{R}^n$-valued functions. It is defined by

$$(S^*\varphi)_i = D_j \varphi_{ij}\ ,$$

and it satisfies

$$\int_\Omega Sf \cdot \varphi dx = - \int_\Omega f \cdot S^* \varphi\, dx \tag{1}$$

when either f or φ has compact support. ($Sf.\ \varphi$ and $f \cdot S^*\ \varphi$ are the dot products $Sf_{ij}\ \varphi_{ij}$ and $f_i\,(S^*\ \varphi)_i$, respectively; dx is the euclidean volume element.)

Equation (1) defines Sf and $S^*\ \varphi$ as *distributions* even if f and φ are not differentiable. We are always assuming that f is continuous and φ locally integrable.

3. Invariance properties

In (1) we prefer to regard $\varphi\, dx$ as a matrix-valued measure, so that the pairing

$$< Sf, \varphi dx > = \int_\Omega Sf \cdot \varphi\, dx$$

is between a function and a measure. Similarly, $S^*\,(\varphi dx) = (S^*\varphi)\, dx$ is a vector-valued measure.

Let A be a Möbius transformation. We define the *pull-backs* of vector- and SM_n-valued functions by

$$(A^*f)(x) = (DA)^{-1}f(Ax)$$
$$(A^*\varphi)(x) = (DA)^{-1}\varphi(Ax)DA$$

and for the corresponding measures by

$$A^*(f\,dx) = |\det A|\,{}^tDAf(Ax)\,dx$$
$$A^*(\varphi\,dx) = |\det A|\,(DA)^{-1}\varphi(Ax)DA\,.$$

These definitions are chosen so that the pairings are invariant:

$$< A^*f, A^*gdx > \; = \; < f, gdx >$$
$$< A^*v, A^*\varphi\,dx > \; = \; < v, \varphi\,dx >\,.$$

There is a basic identity

$$S(A^*f)(x) = (DA)^{-1}Sf(Ax)DA \tag{2}$$

which may be expressed as a commutativity relation $SA^* = A^*S$, applicable to functions, but not to measures. It implies the relation $S^*A^* = A^*S^*$, which is valid for measures in the sense that

$$S^*(A^*\varphi\,dx) = A^*(S^*\varphi\,dx)\,, \tag{3}$$

but not for functions. It should be noted that (2) and (3) are true only because A is conformal.

A function is transformed into a measure by multiplication with a fixed invariant measure $\rho\,dx$. The invariance means that $A^*(\rho\,dx) = \rho\,dx$, or $\rho(Ax)\,|\det DA| = \rho(x)$; we assume also that A leaves Ω invariant. In these circumstances it makes sense to consider the operator $S^*\rho S$ which takes f to $S^*[\rho(Sf)dx]$ and commutes with $A^*:(S^*\rho S)A^* = A^*(S^*\rho S)$.

There are three classical cases in which Ω is invariant under a transitive group $G(\Omega)$ of Möbius transformations:

(i) $\Omega = \mathbf{R}^n$. $G(\Omega)$ is the group of euclidean motions, and $\rho = 1$.

(ii) $\Omega = \mathrm{B}(1) = \{x : |x| < 1\}$. $G = G(B)$ is the group of non-euclidean motions, and $\rho = (1-|x|^2)^{-n}$.

(iii) Ω is the one-point compactification of $\mathbf{R}^n$, identified with S^n in $\mathbf{R}^{n+1}$. The group is formed by the rotations of the sphere, and $\rho = (1+|x|^2)^{-n}$.

4. Non-euclidean motions

The euclidean case was dealt with in [3]. In the present paper we undertake a more detailed study of the hyperbolic case. The unit ball in $\mathbf{R}^n$ is denoted by B, and G is the full group of Möbius transformations mapping B on itself. The Poincaré metric $ds = (1-|x|^2)^{-1} \mid dx \mid$ and the non-euclidean volume element $\rho \, dx = (1-|x|^2)^{-n} \, dx$ are invariant under G.

For $A \in G$ we prefer to denote the Jacobian by $A'(x)$ rather than $DA(x)$. We use $| A'(x) |$ for the linear rate of change, the same in all directions. This notation has the advantage of leading to formulas which are easily recognizable generalizations of the familiar formulas for $n = 2$ in complex notation. $| A'(x) |$ is also the square norm of the matrix $A'(x)$, and $| \det A'(x) | = | A'(x) |^n$.

Reflection in the unit sphere is denoted by $x^* = x/| x |^2$. Its Jacobian is $Dx^* = | x |^{-2} (1_n - 2Q(x))$ with $Q(x)_{ij} = x_i x_j / | x |^2$; note that $(1_n - 2\,Q(x))^2 = 1_n$.

For every $y \in B$ there is a unique $T_y \in G$ such that $T_y y = 0$ and $T'_y(y) = | T'_y(y) | \cdot 1_n$. The most general $A \in G$ is of the form $A = UT_y$ with $y = A^{-1}(0)$ and $U \in O(n)$.

For $n = 2$, in complex notation,

$$T_y x = \frac{x - y}{1 - \bar{y}x}$$

$$T'_y(x) = \frac{1 - | y |^2}{(1 - \bar{y}x)^2} .$$

The first formula can be rewritten as

$$T_y x = \frac{(x - y)(1 - | y |^2) - | x - y |^2 \, y}{| y |^2 \, | x - y^* |^2} .$$

In this form it makes sense for arbitrary n and is in fact the correct formula. The denominator $| y |^2 \, | x - y^* |^2$ corresponds to $| 1 - \bar{y}x |^2$, and it is equal to $1 - 2(xy) + | x |^2 \, | y |^2$, where (xy) is the inner product. To emphasize the symmetry we shall use the notation $| y | \, | x - y^* | = | x | \, | y - x^* | = [x, y]$.

The expression for $T'_y(x)$ is

$$T'_y(x) = \frac{1 - | y |^2}{[x, y]^2} \, \Delta(x, y)$$

with

$$\Delta(x, y) = (1 - 2Q(y))(1 - 2Q(x - y^*)) = (1 - 2Q(y - x^*))(1 - 2Q(x)) .$$

Observe that $\Delta(x, y) = {}^t\Delta(y, x)$ and $\Delta(x, y)^2 = 1_n$ so that $\Delta(x, y) \in O(n)$. The matrix $\Delta(x, y)$ generalizes the angle $\arg(1 - \bar{x}y)/(1 - \bar{y}x)$.

It is useful to note that $|Ax - Ay|^2 = |A'(x)| \, |A'(y)| \, |x - y|^2$ for any Möbius transformation A, and $[Ax, Ay]^2 = |A'(x)| \, |A'(y)| \, [x, y]^2$ if $A \in G$. There is an important relation between $T_y x$ and $T_x y$ expressed by

$$(4) \qquad T_y x = -\Delta(x, y) \, T_x y .$$

We refer to [2, 3, 4, 5] for the elementary proofs of these formulas.

5. Fundamental solutions

A continuous mapping $f : B \to \mathbf{R}^n$ will be called a *deformation*. In this paper we shall assume, mainly for simplicity, that f is continuous on the boundary $S(1)$, and that $x \cdot f(x) = 0$ on $S(1)$; this means that f maps B on itself when regarded as an infinitesimal mapping.

A deformation is *trivial* if $Sf = 0$. There are very few trivial deformations: a complete list is given in [3].

It is customary to say that f is a *quasiconformal* deformation if $\| Sf \| \in L^\infty(B)$; here $\| Sf \|$ is the function whose value at x is the square norm of the matrix $Sf(x)$. More generally, we shall also consider functions with $\| Sf \| \in L^p(B)$; we abbreviate to $Sf \in L^p$, and we denote the L^p-norm of the square norm by $\| Sf \|_p$. The same convention will prevail for all matrix-valued functions.

We shall say that f is *harmonic* if $S^* \rho \, Sf = 0$, $\rho = (1 - |x|^2)^{-n}$. Because of the invariance, if f is harmonic and $A \in G$, then $A^* f$ is also harmonic. Harmonicity in this sense is not the same as requiring the components to be harmonic with respect to the Poincaré metric.

There are n linearly independent solutions of the equation $S^* \gamma = 0$ which are homogeneous of degree $1 - n$. We denote them by $\gamma_{..,k}$, $k = 1, \ldots, n$, the elements being

$$\gamma_{ij,k}(x) = |x|^{-n} (\delta_{ik} x_j + \delta_{jk} x_i - \delta_{ij} x_k) + (n - 2) |x|^{-n-2} x_i x_j x_k .$$

There is a unique vector-valued function $g_{.k}(x)$ with components $g_{ik}(x)$ such that $g_{.k}(x) = 0$ for $|x| = 1$ and $\rho \, Sg_{.k} = \gamma_{..,k}$ so that

$S^* \rho \, S g_{.k} = 0$, or more precisely a Dirac distribution concentrated at 0. It is easy to see that $g = g_{ik}$, which we regard as a Green's matrix, will be of the form $g_{ik}(x) = a(|x|)\,\delta_{ik} + b(|x|)\,x_i x_k$; the explicit expressions for $a(r)$ and $b(r)$ are unimportant, except that g is of order $O((1-|x|^2)^{n+1})$ for $|x| \to 1$ and $O(|x|^{-n+2})$ for $x \to 0$ (if $n = 2$ the latter is replaced by $O(\log 1/|x|)$).

If $U \in O(n)$ it is immediate that $g(Ux) = Ug(x)^t U$. If we replace x by $T_x y$ and U by $-\Delta(x, y)$ it follows with the help of (4) that

$$\Delta(y, x)\, g(T_y x) = g(T_x y)\, \Delta(y, x)\,. \tag{5}$$

We now define the Green's matrix with singularity at y by

Definition 1.

$$\begin{aligned} g_{.k}(x, y) &= (1 - |y|^2)(T_y^* g_{.k})(x) = (1 - |y|^2)\, T_y'(x)^{-1} g(T_y x) \\ &= [x, y]^2 \Delta(y, x)\, g(T_y x)\,. \end{aligned} \tag{6}$$

It is clear that $(S^* \rho\, S)_1\, g(x, y) = 0$ (the subscript indicates that the operator applies to the first variable). In view of (5) we can read off the symmetry property

LEMMA 1. $g(x, y) = {}^t g(y, x)$.

This symmetry plays a prominent role in H. Weyl's classical paper [9] which has been a strong inspiration for this work.

If $A \in G$ it is an easy consequence of (6) that

$$g(Ax, Ay) = A'(x)\, g(x, y)^t A'(y)$$

or, in a more suggestive form,

$$A_1^* A_2^*\, g(x, y) = g(x, y)\,,$$

where A_1^* is A^* applied to the first variable and the first index, and similarly for A_2^*.

Next we define

Definition 2.

$$\gamma_{..,k}(x, y) = \rho(x)\, S_1 g_{.k}(x, y) = (1 - |y|^2)\, \rho(x)\, (S_1 T_y^* g_{.k})(x).$$

It is evident by invariance that $S_1^* \gamma_{..,k}(x, y) = 0$. When x and y are transformed by the same $A \in G$ one finds

$$A_1^* A_2^*\, \gamma_{..,.}(x, y)\, dx = \gamma_{..,.}(x, y)\, dx$$

where A_1^* acts on x and the double index, A_2^* on y and the single index. For $A = T_y$ this leads to the explicit formula

$$\gamma_{..,k}(x, y) = \frac{(1 - |y|^2)^{n+1}}{[x, y]^{2n}} \Delta(y, x) \gamma_{..,k}(T_y x) \Delta(x, y) .$$

We note that $\gamma_{..,.}(x, 0) = \gamma_{..,.}(x)$ and $\gamma_{..,.}(0, y) = -(1-|y|^2)^{n+1} \gamma_{..,.}(y)$.

We shall need to apply S to either variable in $\gamma_{..,.}(x, y)$. For this purpose we introduce

Definition 3. $\Gamma_{ij,hk}(x, y) = [S_2 \gamma_{ij,.}(x, y)]_{hk}$.

Because differentiations with respect to x and y commute it is clear that $S_1^* \Gamma_{..,hk}(x, y) = 0$. Moreover, starting from the relation $g_{ik}(x, y) = g_{ki}(y, x)$ it is not difficult to derive the following symmetry property:

LEMMA 2. $\rho(y) \Gamma_{ij,hk}(x, y) = \rho(x) \Gamma_{hk,ij}(y, x)$.

It follows, in particular, that $S_2^* \rho(y) \Gamma_{ij,..}(x, y) = 0$.

It is also important to know the asymptotic behavior of $\Gamma_{ij,hk}(x, y)$ when $x - y \to 0$. We observe first that

$$\begin{aligned} \rho(y) \Gamma_{ij,hk}(0, y) &= -(1 - |y|^2)^{-n} [S(1 - |y|^2)^{n+1} \gamma_{ij,.}(y)]_{hk} \\ &= - S_{ij,hk}(y) + R_{ij,hk}(y) \end{aligned}$$

where $S_{ij,hk}(y) = [S \gamma_{ij,.}(y)]_{hk}$ is homogeneous of degree $-n$ and $R_{ij,hk}(y)$ is homogeneous of degree $2 - n$. The explicit expression for $\Gamma_{ij,hk}(x, y)$ reads

$$\Gamma_{ij,..}(x, y) = \frac{(1 - |y|^2)^n}{[x, y]^{2n}} \Delta(x, y) \Gamma_{ij,..}(0, T_x y) \Delta(y, x) .$$

Elementary estimates show that

$$(7) \qquad |\Gamma_{ij,hk}(x, y) + S_{ij,hk}(x - y)| \leqq C_n |x - y|^{1-n} [x, y]^{-1}$$

with constant C_n.

6. Potentials

Given an SM_n-valued function v on B we define its *potential* as the vector-valued function $I v$ with components

$$I v(y)_k = \int_B v_{ij}(x) \gamma_{ij,k}(x, y) \, dx .$$

The integral converges if $v \in L^p(B)$ for some p with $n < p \leqq \infty$. In fact, one proves that

$$|Iv(y)| \leqq C_{n,p} \| v \|_p (1 - |y|)^{1-n/p}$$

if $p < \infty$ and

$$|Iv(y)| \leqq C_n \| v \|_\infty (1 - |y|)(1 + \log 1/|(1 - |y|)$$

if $p = \infty$. In any event $Iv(y)$ vanishes at a fixed rate for $|y| \to 1$.

The forming of the potential is an invariant operation in the sense that $IA^*v = A^*Iv$ for every $A \in G$. The potential is harmonic outside the support of v, for $(S^* \rho S)_2 \gamma_{ij,.}(x, y) = 0$.

The following theorem serves to recover f from Sf and its boundary values:

THEOREM 1. *If* $Sf \in L^p(B), p > n,$ *then*

$$c_n f(y) = -ISf(y) + c_n Hf(y) \tag{8}$$

with

$$Hf(y) = \frac{1}{c_n} \int_{S(1)} \gamma_{ij,.}(x, y)\, x_j f_i d\sigma(x) .$$

Moreover, Hf *is the unique harmonic function with the same boundary values as* f, *and if* $x \cdot f = 0$ *on* $S(1)$ *it can also be written in the form*

$$Hf(y) = \frac{1}{c_n} \int_{S(1)} \frac{(1 - |y|^2)^{n+1}}{|x - y|^{2n}} \Delta(x, y) f(x)\, d\sigma(x) .$$

Remarks. $d\sigma$ refers to the $(n-1)$-dimensional measure on $S(1)$, and $c_n = 2(n-1)\,\omega_n/n$ where ω_n is the total measure of $S(1)$. We are assuming that f has a continuous extension to $S(1)$. Actually, this is automatically true if we assume the side condition in the form $x \cdot f(x) \to 0$ as $|x| \to 1$, for it can be shown that $Sf \in L^p$ forces f to satisfy a uniform Hölder condition.

The proof is a straight-forward application of Stokes' formula. The passage from the differentiable to the distributional case is elementary. The fact that a harmonic function is uniquely determined by its boundary values can be demonstrated as follows: Suppose that f is harmonic and zero on $S(1)$. It is readily shown that

$$\int_{S(r)} Sf(x)_{ij}\, \gamma_{ij,k}(x)\, d\sigma = 0$$

for all r. Therefore $ISf(0) = 0$ and hence $f(0) = 0$ by (8). If this result is applied to $(T_y^{-1})^* f$ it follows that $f(y) = 0$ for arbitrary y, so that f is indeed identically zero.

7. Computation of SIv

It is easy to show that $S_{ij,hk}(y) = [S\gamma_{ij,.}(y)]_{hk}$ is a Calderon-Zygmund kernel for any choice of the indices; in other words, it is homogeneous of degree $-n$, and its mean-value over the unit sphere is 0. If $v \in L^p$, $1 < p < \infty$, it follows by the Calderon-Zygmund theory that the principal value

$$\text{pr. v.} \int_B v_{ij}(x)\, S_{ij,hk}(x-y)\, dx$$

exists almost everywhere, and that it is the limit in $L^p(B)$ of the corresponding truncated integrals. In view of (7) it follows that the integral

$$\Gamma v(y)_{hk} = \int_B v_{ij}(x)\, \Gamma_{ij,hk}(x,y)\, dx \tag{9}$$

will also exist as a principal value almost everywhere. One finds, however, that the remainder in (7) makes it possible to assert merely that the principal value is a limit in $L^{p'}$ for any $p' < p/n$. In these circumstances it is natural to assume that $v \in L^p(B)$ for all $p \geqq 1$.

Theorem 2. *If $v \in L^p(B)$ with $p > n$, then $SIv \in L^{p'}(B)$ for all $1 \leqslant p' < p/n$, and*

$$SI\,v = -b_n v + \Gamma v \tag{10}$$

where $b_n = 4\,\omega_n/(n+2)$ and Γv is defined by (9).

Proof. Let φ be an SM_n-valued test-function. The definition of SIv as a distribution leads to the following formal computation:

$$\begin{aligned}
\int_B SI\,v(y)_{hk}\,\varphi(y)_{hk}\,dy &= -\int_B Iv(y)_k S^*\varphi(y)_k\,dy \\
&= -\int_B S^*\varphi(y)_k\,dy \int_B v_{ij}(x)\,\gamma_{ij,k}(x,y)\,dx \\
&= -\int_B v_{ij}(x)\,dx \int_B S^*\varphi(y)_k\,\gamma_{ij,k}(x,y)\,dy \\
&= -\int_B v_{ij}(x)\,dx\,\Big[b_n\,\varphi_{ij}(x) - \int_B \varphi(y)_{hk}\,\Gamma_{ij,hk}(x,y)\,dy\Big].
\end{aligned}$$

The justification, by means of the Zygmund-Calderon theory, is routine, and (10) follows.

Taken together, Theorems 1 and 2 lead to a very striking result:

THEOREM 3. *An SM_n-valued function $v \in L^p(B)$, $p > n$, is of the form $v = Sf$ with $f = 0$ on $S(1)$ if and only if it satisfies the homogeneous integral equation $\Gamma v = -a_n v$ with $a_n = c_n - b_n = 2(n-2)(n+1)\omega_n / n(n+2)$.*

Indeed, if v is of this form, Theorem 1 implies $c_n f = -Iv$, hence $c_n v = -SIv$, and consequently $\Gamma v = (b_n - c_n)v$ by Theorem 2. Conversely, if $\Gamma v = -a_n v$ then $SIv = -c_n v$ by (10), and $f = Iv$ vanishes on $S(1)$.

The point of Theorem 3 is that the solvability of $Sf = v$ (with an extra condition on f) has been reduced to an integral equation.

THEOREM 4. *For any $v \in L^p(B)$, $p > n$, $S^* \rho\, [\Gamma v + a_n v] = 0$.*

Proof. Let f be a vector-valued test-function. Theorem 3 applies to Sf, and we obtain by use of Lemma 2

$$\int_B S^* \rho\, \Gamma v \cdot f\, dx = - \int_B \rho(x)\, \Gamma v(x)_{ij}\, Sf(x)_{ij}\, dx$$

$$= - \int_B \rho(x)\, Sf(x)_{ij}\, dx \int_B v(y)_{hk}\, \Gamma_{hk,ij}(y, x)\, dy$$

$$= - \int_B \rho(y)\, v(y)_{hk} dy \int_B Sf(x)_{ij}\, \Gamma_{ij,hk}(x, y)\, dx$$

$$= - \int_B \rho(y)\, v(y)_{hk}\, \Gamma Sf(y)_{hk} dy = a_n \int_B \rho(y)\, v(y)_{hk}\, Sf(y)_{hk} dy$$

$$= - a_n \int_B S^* \rho v \cdot f\, dy$$

and hence $S^* \rho\, \Gamma v = - a_n S^* v$.

THEOREM 5. *Every v which is in all $L^p(B)$ has a unique representation in the form $v = v' + v''$ where v' and v'' are in all $L^p(B)$ while v' is in the image of SI and v'' is in the kernel of $S^* \rho$.*

As a consequence of Theorems 3 and 4 the representation is given by

$$c_n v = - SIv + (\Gamma v + a_n v).$$

It is unique, for if $SI = \Gamma v + a_n v$, then $S^* \rho\, SIv = 0$ so that Iv is harmonic and 0 on $S(1)$, hence identically zero.

8. Automorphic functions and Beltrami differentials

Although this aspect has not been emphasized it should be clear that the author is trying to develop a theory which is immediately applicable to the study of discrete subgroups of G. All the definitions have been chosen with this in mind, and the relevant theorems for subgroups follow effortlessly.

Let G^0 be a discrete subgroup of G. A vector-valued function f is *automorphic* with respect to G^0 if $A^* f = f$, or more explicitly $A'(x)^{-1} f(Ax) = f(x)$ for all $A \in G^0$. Similarly, an SM_n-valued function v will be called a *Beltrami differential* for G^0 if $A^* v = v$, or $A'(x)^{-1} v(Ax) A'(x) = v(x)$, for all $A \in G^0$. If v is a Beltrami differential, then $A^*(\rho\, v\, dx) = \rho\, v\, dx$ for all $A \in G^0$, and $\rho\, v\, dx$ is called an nth order differential. The terminology is borrowed from the corresponding notions for $n = 2$.

If v is Beltrami and in L^∞, then it is also in $L^p(B)$ for all p, and Theorems 2-5 are applicable. They gain added significance from the fact that Iv is automatically automorphic with respect to G^0 (it is easy to show that $A^* Iv = IA^* v$ for all v and $A \in G$). As a consequence SIv is Beltrami, and by Theorem 2 the same is true of Γv. It follows that Theorems 2-5 may be interpreted as referring to the quotient space $G^0 \backslash B$, provided that we start from the hypothesis $v \in L^\infty$. In the conclusion we know, for instance, that

$$\int_B \| SI\gamma \|^p dx = \int_{G^0 \backslash B} \| SI\, v \|^p \rho_0 dx < \infty$$

where, by a theorem of Godement,

$$\rho_0(x) = \sum_{A \in G^0} | A'(x) |^n$$

is known to converge.

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