

## 2. Definitions and notations

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# A SINGULAR INTEGRAL EQUATION CONNECTED WITH QUASICONFORMAL MAPPINGS IN SPACE

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*Dedicated to Albert Pfluger for his seventieth birthday*

## 1. INTRODUCTION

This paper continues the author's investigation of two differential operators,  $S$  and  $S^*$ , which arise naturally in the study of infinitesimal quasiconformal mappings in  $n$  dimensions (see References). If  $\Omega$  is open in  $\mathbf{R}^n$  the operator  $S$  acts on functions  $f : \Omega \rightarrow \mathbf{R}^n$  and has values  $Sf \in SM_n$ , where  $SM_n$  is the space of symmetric  $n \times n$  matrices with zero trace. Definitions are in Sec. 2.

A key question is the solvability of the inhomogeneous equation  $Sf = v$ . For  $n = 2$ ,  $Sf$  can be identified with the complex derivative  $f_{\bar{z}}$  of a complex-valued function, and the problem is that of recovering  $f$  from  $f_{\bar{z}}$ . As well known, this problem has always a solution, and it is given by the generalized Cauchy formula, also known as Pompeiu's formula. For  $n > 2$  the right hand member  $v$ , an  $SM_n$ -valued function, must satisfy certain conditions, which are known in principle, as limiting cases of the Weyl-Schouten conditions of vanishing conformal curvature.

These conditions, although explicit, are quite intractable. It is therefore rather surprising that a necessary and sufficient condition for  $Sf = v$  to be solvable can be expressed as a singular homogeneous integral equation satisfied by  $v$ . This integral equation can be treated by the methods of Calderon and Zygmund.

## 2. DEFINITIONS AND NOTATIONS

A quasiconformal homeomorphism  $F : \Omega \rightarrow F(\Omega)$  is known to be differentiable almost everywhere. We denote its Jacobian matrix by  $DF$ . The normalized Jacobian is  $XF = (\det DF)^{-1/n} DF$ , and  $MF = {}^tXF \cdot XF$

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is the normalized and symmetrized Jacobian; it carries the quasiconformal data of the mapping.

The Riemannian metric  $ds^2 = {}^t dx (MF) dx$  is conformally flat, a condition expressed by the vanishing of the conformal curvature tensor. For  $n = 3$  this tensor is identically zero, but there is instead an integrability condition.

Let  $F(x, t)$  be a one-parameter family of homeomorphisms such that  $F(x, 0) = x$ ,  $\dot{F}(x, 0) = f(x)$ . Under suitable regularity conditions  $(DF)_0 = Df$ ,  $(XF)_0 = Df - \frac{1}{n} \text{tr } Df \cdot 1_n$ , and  $(MF)_0 = Df + {}^t Df - \frac{2}{n} \text{tr } Df \cdot 1_n$ . This motivates introducing the differential operator  $S$  defined by

$$(Sf)_{ij} = \frac{1}{2} (D_i f_j + D_j f_i) - \frac{1}{n} \delta_{ij} D_k f_k.$$

(The summation convention is in force in this paper). Note that  $Sf$  has values in  $SM_n$ .

There is a formal adjoint  $S^*$  which maps  $SM_n$ -valued functions on  $\mathbf{R}^n$ -valued functions. It is defined by

$$(S^* \varphi)_i = D_j \varphi_{ij},$$

and it satisfies

$$(1) \quad \int_{\Omega} Sf \cdot \varphi dx = - \int_{\Omega} f \cdot S^* \varphi dx$$

when either  $f$  or  $\varphi$  has compact support. ( $Sf \cdot \varphi$  and  $f \cdot S^* \varphi$  are the dot products  $Sf_{ij} \varphi_{ij}$  and  $f_i (S^* \varphi)_i$ , respectively;  $dx$  is the euclidean volume element.)

Equation (1) defines  $Sf$  and  $S^* \varphi$  as *distributions* even if  $f$  and  $\varphi$  are not differentiable. We are always assuming that  $f$  is continuous and  $\varphi$  locally integrable.

### 3. INVARIANCE PROPERTIES

In (1) we prefer to regard  $\varphi dx$  as a matrix-valued measure, so that the pairing

$$\langle Sf, \varphi dx \rangle = \int_{\Omega} Sf \cdot \varphi dx$$

is between a function and a measure. Similarly,  $S^*(\varphi dx) = (S^* \varphi) dx$  is a vector-valued measure.