

# 6. POTENTIALS

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where  $A_1^*$  acts on  $x$  and the double index,  $A_2^*$  on  $y$  and the single index. For  $A = T_y$  this leads to the explicit formula

$$\gamma_{\dots,k}(x, y) = \frac{(1 - |y|^2)^{n+1}}{[x, y]^{2n}} \Delta(y, x) \gamma_{\dots,k}(T_y x) \Delta(x, y).$$

We note that  $\gamma_{\dots}(x, 0) = \gamma_{\dots}(x)$  and  $\gamma_{\dots}(0, y) = -(1 - |y|^2)^{n+1} \gamma_{\dots}(y)$ .

We shall need to apply  $S$  to either variable in  $\gamma_{\dots}(x, y)$ . For this purpose we introduce

*Definition 3.*  $\Gamma_{ij,hk}(x, y) = [S_2 \gamma_{ij,\dots}(x, y)]_{hk}$ .

Because differentiations with respect to  $x$  and  $y$  commute it is clear that  $S_1^* \Gamma_{\dots,hk}(x, y) = 0$ . Moreover, starting from the relation  $g_{ik}(x, y) = g_{ki}(y, x)$  it is not difficult to derive the following symmetry property:

LEMMA 2.  $\rho(y) \Gamma_{ij,hk}(x, y) = \rho(x) \Gamma_{hk,ij}(y, x)$ .

It follows, in particular, that  $S_2^* \rho(y) \Gamma_{ij,\dots}(x, y) = 0$ .

It is also important to know the asymptotic behavior of  $\Gamma_{ij,hk}(x, y)$  when  $x - y \rightarrow 0$ . We observe first that

$$\begin{aligned} \rho(y) \Gamma_{ij,hk}(0, y) &= -(1 - |y|^2)^{-n} [S(1 - |y|^2)^{n+1} \gamma_{ij,\dots}(y)]_{hk} \\ &= -S_{ij,hk}(y) + R_{ij,hk}(y) \end{aligned}$$

where  $S_{ij,hk}(y) = [S \gamma_{ij,\dots}(y)]_{hk}$  is homogeneous of degree  $-n$  and  $R_{ij,hk}(y)$  is homogeneous of degree  $2 - n$ . The explicit expression for  $\Gamma_{ij,hk}(x, y)$  reads

$$\Gamma_{ij,\dots}(x, y) = \frac{(1 - |y|^2)^n}{[x, y]^{2n}} \Delta(x, y) \Gamma_{ij,\dots}(0, T_x y) \Delta(y, x).$$

Elementary estimates show that

$$(7) \quad |\Gamma_{ij,hk}(x, y) + S_{ij,hk}(x - y)| \leq C_n |x - y|^{1-n} [x, y]^{-1}$$

with constant  $C_n$ .

## 6. POTENTIALS

Given an  $SM_n$ -valued function  $v$  on  $B$  we define its *potential* as the vector-valued function  $Iv$  with components

$$Iv(y)_k = \int_B v_{ij}(x) \gamma_{ij,k}(x, y) dx.$$

The integral converges if  $v \in L^p(B)$  for some  $p$  with  $n < p \leq \infty$ . In fact, one proves that

$$|Iv(y)| \leq C_{n,p} \|v\|_p (1 - |y|)^{1-n/p}$$

if  $p < \infty$  and

$$|Iv(y)| \leq C_n \|v\|_\infty (1 - |y|)(1 + \log 1/(1 - |y|))$$

if  $p = \infty$ . In any event  $Iv(y)$  vanishes at a fixed rate for  $|y| \rightarrow 1$ .

The forming of the potential is an invariant operation in the sense that  $IA^*v = A^*Iv$  for every  $A \in G$ . The potential is harmonic outside the support of  $v$ , for  $(S^* \rho S)_2 \gamma_{ij..}(x, y) = 0$ .

The following theorem serves to recover  $f$  from  $Sf$  and its boundary values:

**THEOREM 1.** *If  $Sf \in L^p(B)$ ,  $p > n$ , then*

$$(8) \quad c_n f(y) = -ISf(y) + c_n Hf(y)$$

with

$$Hf(y) = \frac{1}{c_n} \int_{S(1)} \gamma_{ij..}(x, y) x_j f_i d\sigma(x).$$

Moreover,  $Hf$  is the unique harmonic function with the same boundary values as  $f$ , and if  $x \cdot f = 0$  on  $S(1)$  it can also be written in the form

$$Hf(y) = \frac{1}{c_n} \int_{S(1)} \frac{(1 - |y|^2)^{n+1}}{|x - y|^{2n}} \Delta(x, y) f(x) d\sigma(x).$$

*Remarks.*  $d\sigma$  refers to the  $(n-1)$ -dimensional measure on  $S(1)$ , and  $c_n = 2(n-1)\omega_n/n$  where  $\omega_n$  is the total measure of  $S(1)$ . We are assuming that  $f$  has a continuous extension to  $S(1)$ . Actually, this is automatically true if we assume the side condition in the form  $x \cdot f(x) \rightarrow 0$  as  $|x| \rightarrow 1$ , for it can be shown that  $Sf \in L^p$  forces  $f$  to satisfy a uniform Hölder condition.

The proof is a straight-forward application of Stokes' formula. The passage from the differentiable to the distributional case is elementary. The fact that a harmonic function is uniquely determined by its boundary values can be demonstrated as follows: Suppose that  $f$  is harmonic and zero on  $S(1)$ . It is readily shown that

$$\int_{S(r)} Sf(x)_{ij} \gamma_{ij,k}(x) d\sigma = 0$$

for all  $r$ . Therefore  $ISf(0) = 0$  and hence  $f(0) = 0$  by (8). If this result is applied to  $(T_y^{-1})^* f$  it follows that  $f(y) = 0$  for arbitrary  $y$ , so that  $f$  is indeed identically zero.

### 7. COMPUTATION OF $SIv$

It is easy to show that  $S_{ij,hk}(y) = [S\gamma_{ij,\cdot}(y)]_{hk}$  is a Calderon-Zygmund kernel for any choice of the indices; in other words, it is homogeneous of degree  $-n$ , and its mean-value over the unit sphere is 0. If  $v \in L^p$ ,  $1 < p < \infty$ , it follows by the Calderon-Zygmund theory that the principal value

$$\text{pr. v. } \int_B v_{ij}(x) S_{ij,hk}(x-y) dx$$

exists almost everywhere, and that it is the limit in  $L^p(B)$  of the corresponding truncated integrals. In view of (7) it follows that the integral

$$(9) \quad \Gamma v(y)_{hk} = \int_B v_{ij}(x) \Gamma_{ij,hk}(x,y) dx$$

will also exist as a principal value almost everywhere. One finds, however, that the remainder in (7) makes it possible to assert merely that the principal value is a limit in  $L^{p'}$  for any  $p' < p/n$ . In these circumstances it is natural to assume that  $v \in L^p(B)$  for all  $p \geq 1$ .

**THEOREM 2.** *If  $v \in L^p(B)$  with  $p > n$ , then  $SIv \in L^{p'}(B)$  for all  $1 \leq p' < p/n$ , and*

$$(10) \quad SIv = -b_n v + \Gamma v$$

where  $b_n = 4\omega_n/(n+2)$  and  $\Gamma v$  is defined by (9).

*Proof.* Let  $\varphi$  be an  $SM_n$ -valued test-function. The definition of  $SIv$  as a distribution leads to the following formal computation:

$$\begin{aligned} \int_B SIv(y)_{hk} \varphi(y)_{hk} dy &= - \int_B Iv(y)_k S^* \varphi(y)_k dy \\ &= - \int_B S^* \varphi(y)_k dy \int_B v_{ij}(x) \gamma_{ij,k}(x,y) dx \\ &= - \int_B v_{ij}(x) dx \int_B S^* \varphi(y)_k \gamma_{ij,k}(x,y) dy \\ &= - \int_B v_{ij}(x) dx [b_n \varphi_{ij}(x) - \int_B \varphi(y)_{hk} \Gamma_{ij,hk}(x,y) dy]. \end{aligned}$$