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INVARIANTS OF FINITE REFLECTION GROUPS

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INTRODUCTION

Let G be a group of linear transformations acting on a finite dimensional vector space V over a given field k. Let S be the ring of polynomial functions on V, i.e. those functions which become polynomials for any given co-ordinate system on V. G is made to act on S by defining

$$(\sigma s)(v) = s(\sigma^{-1} v), \qquad \sigma \in G,\ s \in S,\ v \in V$$

The elements of S fixed by G, i.e. $\sigma s = s$ for all $\sigma \in G$, are called the invariants of G. The subject of invariant theory deals with the determination of all invariants of a given group G. For finite groups, Hilbert proved in 1890 [14] the main theorem of invariant theory stating that the algebra of invariants is finitely generated. These finite sets of generators are said to form an integrity basis for the invariants of G. Later on, Noether [17] produced an explicit set of basic invariants for finite groups. However, this number is usually much more than necessary (we elaborate on this point in chapter I) and there lacks a systematic method for producing a basis which is in some sense minimal.

As we show in this expository paper, such a systematic method exists for the class of groups known as the finite reflection groups. In this case, a very detailed and beautiful theory has been worked out in the last twenty five years, bringing together various concepts from algebra, geometry, and analysis. The subject matter is closely related to other mathematical theories, such as the topology of Lie groups and the study of the Chevalley groups. For these connections, the interested reader is referred to the books of Bourbaki and Carter [2, 3], where further references are supplied.

We give here a brief description of the subject treated in this paper. A linear transformation σ acting on the n-dimensional vector space V is said to be a reflection if it fixes an $n - 1$ dimensional hyperplane π, which is then called the reflecting hyperplane (r.h.) of σ. G is a reflection group if it is generated by reflections. For finite reflection groups G acting on an

n-dimensional vector space V over a field k of characteristic 0, we have the fundamental result of Chevalley [4], stating that there are n algebraically independent homogeneous polynomials forming an integrity basis for the invariants of G. Conversely, we will show that if G is a finite group of linear transformations acting on V which is not a reflection group, than any basic set of homogeneous invariants must contain more than n elements which are algebraically dependent. Thus we may say that the finite reflection groups are distinguished to be those with the simplest possible type of invariant theory.

Let $d_1, \ldots, d_n$ be the respective degrees of the basic homogeneous invariants $I_1, \ldots, I_n$, where $d_1 \leqslant \ldots \leqslant d_n$. It can readily be shown that the d_i's are independent of the particular basis $I_1, \ldots, I_n$. We present in chapter III two methods for computing the d_i's. The first one is due to Coxeter and Coleman [7, 8] and is restricted to the case where the underlying field k is real. Coxeter has classified all real finite irreducible reflection groups [6]. If such a group G acts on the n-dimensional Euclidean space R^n, then its r.h.'s divide R^n into $|G|$ components, called the chambers of G. Each chamber is bounded by n r.h.'s called its walls. The reflections in these walls generate G. Coxeter has found a remarkable relation between the d_i's and the eigenvalues of the product of these generators. This relation, first checked individually for each of the groups listed in [7], has subsequently been proved by Coleman [8]. Coleman's Theorem (Theorem 3.8 of chapter III) may be used effectively to compute the d_i's in the real case. We also present another method due to Solomon [18] who has obtained formula 3.27) for the d_i's. Solomon's method works for all fields of characteristic 0, but cannot be used as effectively as the method of Coxeter and Coleman in the real case.

In Chapter IV, we apply the invariant theory developed in the earlier chapters to study a certain system of partial differential equations and related mean value properties. We assume that G is a finite orthogonal reflection group acting on R^n. Let I denote the set of homogeneous invariants of positive degree. For any polynomial $p(x)$, let $p(\partial)$ be the partial differential operator obtained by replacing each variable x_i by $\partial/\partial x_i$. Steinberg [21] has described the solution space of C^∞ functions satisfying the system

1) $$p(\partial)f = 0, \; p \in I$$

on some given n-dimensional region $\mathscr{R}$. We may interpret the solutions of 1) to be an analog of the harmonic functions, as the latter are the solutions of $\sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2} = 0$ and $\sum_{i=1}^{n} x_i^2$ is the basic invariant for the orthogonal group $O(n)$ ([23] p. 53). We use Steinberg's result to describe the solution space S_y of continuous functions on $\mathscr{R}$ satisfying the mean value property

$$2) \qquad f(x) = \frac{1}{|G|} \sum_{\sigma \varepsilon G} f(x + t\sigma y), \ x \in \mathscr{R}$$

and $0 < t < \varepsilon_x$, y denoting a fixed vector $\neq 0$. Observe that 2) is again an analog of the familiar mean value property characterizing harmonic functions ([15] p. 224). Flatto and Wiener [10] have shown that the solution spaces to 1) and 2) are identical, provided the degrees d_i are distinct and y does not belong to a certain algebraic manifold $\mathscr{M}$. $\mathscr{M}$ can be described by equations, the latter yielding an explicit integrity basis for the invariants of G.

I have tried to keep the present paper self-contained, defining and explaining most of the notions and results needed in it. Occasionally, I quote some well known results of algebra, most of which can be found in [22]. In Chapter IV we require some standard results on harmonic functions, which may be found in [15]. In Chapter III, we require Coxeter's classification of the irreducible finite reflection groups acting on R^n. It would have taken us too far afield to present this matter in detail. I present a brief exposition, without proof, of the main points of this theory which are required in the present paper. For a quick and readable account of the details, the reader is referred to [1].

CHAPTER I

GENERAL THEORY

1. The Main Theorem of Invariant Theory

We present in this chapter some basic notions and results of invariant theory. We assume throughout that G is a finite group of linear transformations acting on the finite dimensional vector space V over a given field k of characteristic 0. n designates the dimension of V.

Definition 1.1. Let $P(v)$ be a polynomial function on V. $P(v)$ is invariant of $G \Leftrightarrow P(\sigma v) = P(v)$ for $\sigma \in G$, $v \in V$.

Let $x_1, \ldots, x_n$ be a coordinate system for V. Then $P(v)$ becomes a polynomial which we designate by $P(x)$. σ is represented by a matrix which we