

2. Molién's Formula

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THEOREM 1.2. *Let I_1, \dots, I_l form a basis for the invariants of G . We may choose from the I_j 's n elements which are algebraically independent over k . Thus $l \geq n$.*

Proof. Let $k(x_1, \dots, x_n)$ be the field of rational functions in the indeterminates x_1, \dots, x_n with coefficients in k , a similar meaning being attached to $k(I_1, \dots, I_l)$. We show that $k(x_1, \dots, x_n)$ is a finite extension of $k(I_1, \dots, I_l)$. Let $x_i(x) = x_i$ and set

$$(1.7) \quad p_i(X) = \prod_{\sigma \in G} (X - x_i(\sigma x)) = X^{|G|-1} + a_1 X^{|G|-2} + a_2 X^{|G|-3} + \dots + a_{|G|}$$

It is readily checked that the coefficients a_j are polynomials which are invariant under G . Thus each $a_j \in k(I_1, \dots, I_l)$. Since $p_i(x_i) = 0$, we conclude that x_i, \dots, x_n are algebraic over $k(I_1, \dots, I_l)$. Hence $k(x_1, \dots, x_n)$ is a finite extension of $k(I_1, \dots, I_l)$.

Let $K = k(\alpha_1, \dots, \alpha_s)$ be the field obtained by adjoining $\alpha_1, \dots, \alpha_s$ to k . We may define the transcendence degree of K over k to be the maximum number of α_i 's which are algebraically independent over k ([22], Vol. 1, p. 201). We denote this degree by $\text{Tr.deg. } K/k$. If we have three fields $k \subset K \subset L$, then it is known that

$$(1.8) \quad \text{Tr.deg. } L/k = \text{Tr.deg. } L/K + \text{Tr.deg. } K/k \text{ ([22], Vol. 1, p. 202).}$$

Apply (1.8) with $L = k(x_1, \dots, x_n)$, $K = k(I_1, \dots, I_l)$. Then $\text{Tr.deg. } L/k = n$ and the finiteness of L over K means that $\text{Tr.deg. } L/K = 0$. Hence $\text{Tr.deg. } K/k = n$, which means that we may choose n I_j 's which are algebraically independent over k .

2. MOLLIEN'S FORMULA

For each integer $m \geq 0$, the homogeneous invariants of degree m form a finite dimensional vector space over k of dimension δ_m . We derive an interesting and useful formula for the δ_m 's.

THEOREM 1.3. (Molien's Formula [16]). *Let $\omega_1(\sigma), \dots, \omega_n(\sigma)$ be the eigenvalues of σ . Then*

$$(1.9) \quad \sum_{m=0}^{\infty} \delta_m t^m = \frac{1}{|G|} \sum_{\sigma \in G} \frac{1}{(1 - \omega_1(\sigma)t) \dots (1 - \omega_n(\sigma)t)}$$

REMARK. (1.9) is to be interpreted as an identity between two formal power series. I.e. if the right side is expanded as a formal power series, then its coefficients are identical with the δ_m 's.

We require the following

LEMMA 1.2. Let W be the subspace fixed by G .

$$\text{Then } \dim W = \frac{1}{|G|} \sum_{\sigma \in G} \text{Tr}(\sigma).$$

Proof. Let $\{v_1, \dots, v_r\}$ be a basis for W and augment this to a basis $\{v_1, \dots, v_n\}$ for V . For $\sigma_1 \in G$ and $v \in V$, we have

$$\sigma_1 \left(\sum_{\sigma \in G} \sigma v \right) = \sum_{\sigma \in G} (\sigma_1 \sigma) v = \sum_{\sigma \in G} \sigma v,$$

so that $\sum_{\sigma \in G} \sigma v \in W$. It follows that

$$\frac{1}{|G|} \sum_{\sigma \in G} \sigma v_i = v_i, \quad 1 \leq i \leq r,$$

and

$$\frac{1}{|G|} \sum_{\sigma \in G} \sigma v_i = \sum_{j=1}^r a_{ij} v_j, \quad r+1 \leq i \leq n,$$

the a_{ij} 's $\in k$. Hence

$$\frac{1}{|G|} \sum_{\sigma \in G} \text{Tr} \sigma = \text{TR} \left(\frac{1}{|G|} \sum_{\sigma \in G} \sigma \right) = r = \dim W.$$

Proof of Theorem 1.3. Let \tilde{k} = algebraic closure of k . For any $\sigma \in G$, we can find a matrix τ with entries in \tilde{k} so that $\tau \sigma \tau^{-1} = d$, d being diagonal and the diagonal entries being the eigenvalues of σ . Let R_m, \tilde{R}_m denote respectively the space of homogeneous polynomials with coefficients from k, \tilde{k} . Let $(\text{Tr} \sigma)_m$ = trace of σ as a transformation on R_m = trace of σ as a transformation on \tilde{R}_m . Let $(\text{Tr} d)_m$ = trace of d as a transformation on \tilde{R}_m . We have $d(P(x)) = P(d^{-1}x)$ for any polynomial $P(x)$. In particular, for any monomial x^a , we have $d(x^a) = \omega^a(\sigma^{-1})$, where $\omega(\sigma) = (\omega_1(\sigma), \dots, \omega_n(\sigma))$. The monomials x^a form a basis for R_m and \tilde{R}_m . We conclude that

$$(1.10) \quad (\text{Tr} \sigma)_m = (\text{Tr} d)_m = \sum_{|a|=m} \omega^a(\sigma^{-1}).$$

(1.10) and Lemma 1.2 yield

$$(1.11) \quad \delta_m = \frac{1}{|G|} \sum_{\sigma \in G} (\text{Tr } \sigma)_m = \frac{1}{|G|} \sum_{\sigma \in G} \sum_{|a|=m} \omega^a(\sigma).$$

Multiply both sides of (1.11) by t^m and sum over m from 0 to ∞ . We get

$$\begin{aligned} \sum_{m=0}^{\infty} \delta_m t^m &= \frac{1}{|G|} \sum_{m=0}^{\infty} \sum_{\sigma \in G} \sum_{|a|=m} \omega^a(\sigma) t^m \\ &= \frac{1}{|G|} \sum_{\sigma \in G} \left\{ \sum_{m=0}^{\infty} \omega_1^m(\sigma) t^m \dots \sum_{m=0}^{\infty} \omega_n^m(\sigma) t^m \right\} \\ &= \frac{1}{|G|} \sum_{\sigma \in G} \frac{1}{(1 - \omega_1(\sigma) t) \dots (1 - \omega_n(\sigma) t)} \end{aligned}$$

CHAPTER II

INVARIANT THEORETIC CHARACTERIZATION OF FINITE REFLECTION GROUPS

1. CHEVALLEY'S THEOREM

We showed in chapter I that we can always find a finite number of homogeneous invariants forming a basis for the invariants of G and that this set must contain at least n elements, where $n = \dim V$. We show that this lower bound is attained only for the finite reflection groups. We first define these groups.

DEFINITION 2.1. Let σ be a linear transformation acting on the n -dimensional vector space V . σ is a reflection $\Leftrightarrow \sigma$ fixes an $n - 1$ dimensional hyperplane π and σ is of finite order > 1 . π is called the reflecting hyperplane (r.h.) of σ .

REMARK. Choose $v \notin \pi$. and let $\sigma v = \zeta v + p$, $p \in \pi$. If $\zeta = 1$, then $\sigma^m v = v + mp$, contradicting that σ is of finite order. Hence $\zeta \neq 1$. Let $v' = v + (\zeta - 1)^{-1} p$ and choose p_1, \dots, p_{n-1} as a basis for π . Then $\sigma p_i = p_i$, $1 \leq i \leq n - 1$, $\sigma v' = \zeta v'$. ζ is a root of 1 in k which is distinct from 1, as σ is of finite order > 1 . Thus σ is a reflection iff relative to some basis, the matrix for σ is diagonal, $n - 1$ of the diagonal entries equalling 1 and the remaining one equalling a root of 1 in k distinct from 1.