1. The Classification of the Finite Real Reflection Groups

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **24 (1978)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **13.09.2024**

degrees. The second method (Theorem 3.14) is valid for an arbitrary field of characteristic 0, but is less effective than the first in the real case.

We first prove that the degrees of the basic invariants are independent of any particular basis.

THEOREM 3.1. *Let G a finite reflection group acting on the n-dimensional vector space V. Let $I_1, \ldots, I_n$ be homogeneous polynomials of respective degrees $d_1 \leqslant \ldots \leqslant d_n$ forming a basis for the invariants of G. $d_1, \ldots, d_n$ are independent of the chosen basis $I_1, \ldots, I_n$.*

Proof. Let $J_1, \ldots, J_n$ be another set of homogeneous invariants forming a basis for the invariants of G. Let $d_1' \leqslant \ldots \leqslant d_n'$ be the respective degrees of $J_1, \ldots, J_n$. We must show that $d_i' = d_i$, $1 \leqslant i \leqslant n$. If not, then let i_0 be the smallest i such that $d_{i_0}' \neq d_{i_0}$, say $d_{i_0}' < d_{i_0}$. Each J_i is a polynomial in those I_i's whose degree $\leqslant \deg J_i$. It follows that for $1 \leqslant i \leqslant i_0$, $J_i = P_i(I_1, \ldots, I_{i_0-1})$, $P_i(y_1, \ldots, y_{i_0-1})$ being a polynomial in $y_1, \ldots, y_{i_0-1}$. Hence $J_1, \ldots, J_{i_0}$ are algebraically dependent over k ([22], Vol. 1, p. 181), contradicting that $J_1, \ldots, J_n$ are algebraically independent over k (Theorem 1.2). Thus $d_i' = d_i$, $1 \leqslant i \leqslant n$.

Theorem 3.1. shows that the numbers $d_1, \ldots, d_n$ are determined by G. We shall give an effective method for the computation of the d_i's in case the underlying field k is real. We first digress to discuss the classification of the finite real reflection groups.

1. The Classification of the Finite Real Reflection Groups

These groups have been classified by Coxeter [6]. We give here a brief description of the theory, as we require it for the computation of the d_i's.

We first observe that we may assume G to be orthogonal.

THEOREM 3.2. *Let G be a finite group acting on the n-dimensional Euclidean space R^n. There exists a non-singular transformation τ on R^n such that the group $\tau^{-1} G \tau$ consists of orthogonal transformations.*

Proof. Let $P(x) = \sum_{\sigma \varepsilon G} (\sigma x, \sigma x)$ where $x = (x_1, \ldots, x_n)$ and (x, y) is the inner product of x and y. For $x \neq 0$, each $(\sigma x, \sigma x) > 0$ so that $P(x) > 0$. Furthermore for $\sigma_1 \in G$, $P(\sigma_1 x) = \sum_{\sigma \varepsilon G} (\sigma \sigma_1 x, \sigma \sigma_1 x)$ $= \sum_{\sigma \varepsilon G} (\sigma x, \sigma x) = P(x)$. Thus $P(x)$ is a positive definite quadratic form invariant under G. Choose $x = \tau y$ so that $P(\tau y) = (y, y)$. We have $(\tau^{-1}\sigma\tau y, \tau^{-1}\sigma\tau y) = P(\sigma\tau y) = P(\tau y) = (y, y)$, $\sigma \in G$, so that the transformations $\tau^{-1}\sigma\tau$ are orthogonal.

Thus all transformations of G become orthogonal after a suitable linear change of variables. We assume from now on that G is orthogonal. If G is a finite reflection group, this condition is equivalent to demanding that all reflections of G are orthogonal. I.e. for any reflection σ, σ fixes all vectors in the r.h. π and $\sigma(v) = -v$, iff v is perpendicular to π. The two unit vectors perpendicular to π are called roots of G. The set of all roots is called the root system of G.

DEFINITION 3.1. Let F be a region of R^n, G a finite group acting on R^n. F is a fundamental region for G iff:

i) $\sigma_1 F \cap \sigma_2 F = \Phi$ whenever $\sigma_1 \neq \sigma_2$,

ii) $R^n = \bigcup_{\sigma \varepsilon G} \sigma \bar{F}$, $\bar{F}$ being the closure of F.

We remark that it suffices to know i) for $\sigma_1 = e$, the identity of G. For $\sigma_1 F \cap \sigma_2 F = \Phi$ iff $\sigma_1^{-1}(\sigma_1 F \cap \sigma_2 F) = F \cap \sigma_1^{-1}\sigma_2 F = \Phi$. If F is a fundamental region, then so is σF, $\sigma \in G$. The group G permutes these fundamental regions and acts transitively on them.

THEOREM 3.3. *Let G be a finite reflection group acting on R^n. Assume that the roots of G span R^n (G is then called a Coxeter group). The complement of the union of the r.h.'s of G consist of $|G|$ fundamental regions called the chambers of G. G permutes these chambers and acts transitively on them. Each chamber F is bounded by n r.h.'s called the walls of F. Let $r_1, \ldots, r_n$ be the n roots perpendicular to the n walls $W_1, \ldots, W_n$ and pointing into F, and let R_i be the reflection in W_i. The r_i's are linearly independent and $r_i \cdot r_j = -\cos \pi/p_{ij}$, $p_{ii} = 1$ and p_{ij} being an integer $\geqslant 2$ if $i \neq j$. The R_i's generate G.*

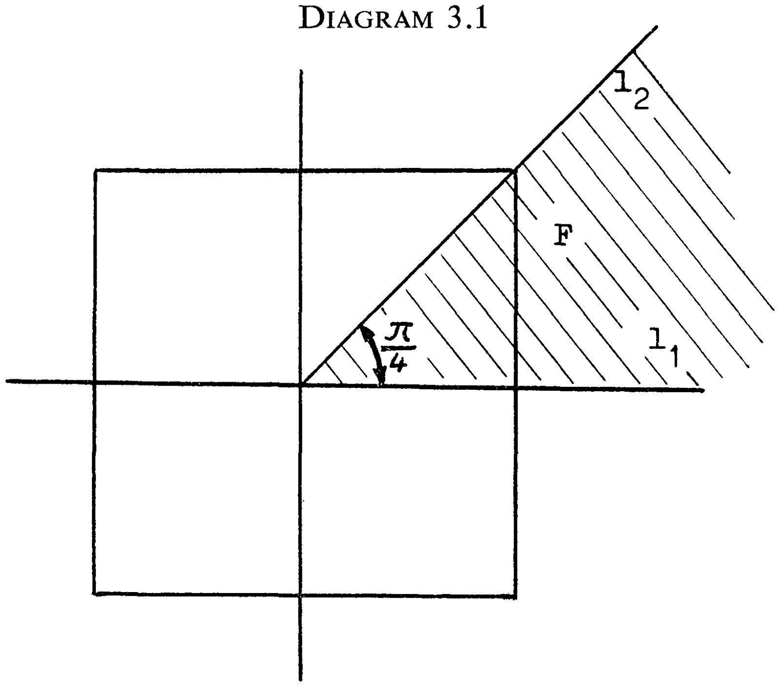
We have $F = \{x \mid x \cdot r_i > 0, 1 \leqslant i \leqslant n\}$. F may also be described as follows. Choose $\{r_1', \ldots, r_n'\}$ to be the dual basis to $\{r_1, \ldots, r_n\}$; i.e. $(r_i, r_j') = \delta_{ij}$. For any x, $x = \sum_{i=1}^{n} (x \cdot r_i) r_i'$. Thus

$$F = \{x \mid x = \sum_{i=1}^{n} \lambda_i r_i', \lambda_i > 0 \text{ for } 1 \leqslant i \leqslant n\}.$$

F is thus a wedge with n walls, the vectors r_i' lying along its edges. The angle between the walls W_i, W_j $(i \neq j)$ is readily seen to be π/p_{ij}. We refer to $\{r_1, ..., r_n\}$ as a fundamental system of roots and to $R_1, ..., R_n$ as a fundamental system of reflections.

As a simple illustration of the above concepts, we choose G to be the group of symmetries of a regular n-gon p_n. G is then called the dihedral group of order $2n$ and we denote it by H_2^n. Assume that the center of the polygon is at the origin. We choose in this case two rays l_1, l_2 emanating from the origin making an angle π/n, one of the rays passing through a vertex of p_n, the other through a mid-point of a side of p_n (see the diagram where $n = 4$). F is the wedge with sides l_1, l_2. The reflections in l_1, l_2 generate H_2^n.

DIAGRAM 3.1



For any Coxeter group G acting on R^n, we introduce the associated Coxeter graph $\mathcal{G}$ as follows. Let $\mathcal{G}$ consist of n points, called the nodes and label these as $1, ..., n$. We set up the $1 - 1$ correspondence $i \leftrightarrow r_i$, $r_1, ..., r_n$ being the fundamental root system of Theorem 3.3. The i-th and j-th node $(i \neq j)$ are joined by a branch iff $(r_i, r_j) \neq 0$. If this be the case then $p_{ij} \geqslant 3$; we mark the branch joining i to j by p_{ij} whenever $p_{ij} > 3$, and omit a mark if $p_{ij} = 3$. Eg. the graph associated with H_2^n is ○——○ for $n = 3$ and ○—n—○ for $n \geqslant 4$.

The motivation for the rather artificial looking definition of $\mathcal{G}$ stems from the following facts.

THEOREM 3.4. *Let G be a Coxeter group acting on R^n. G is irreducible iff its corresponding graph is connected.*

Proof. If the graph of G has more than one component, then the root system $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$ where $\mathcal{R}_1, \mathcal{R}_2$ are disjoint and non-empty, the roots in $\mathscr{R}_1$ being perpendicular to those in $\mathscr{R}_2$. Let V be the span of the roots in $\mathscr{R}_1$. If σ is a reflection corresponding to a root in $\mathscr{R}_1$, then $\sigma\,|_V$ is a reflection of V. If σ is a reflection corresponding to a root in $\mathscr{R}_2$, then $\sigma\,|_V$ = identity. Since the reflections generate G, V is a proper invariant subspace.

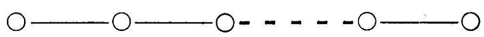
Conversely, let V be a proper invariant subspace of G. Then so is the orthogonal complement $V^{\perp}$. The proof of Theorem 2.7 shows that every root is either in Vor $V^{\perp}$. Since the roots span R^n, there are roots both in V and $V^{\perp}$. Since the roots in $\mathscr{R} \cap V$ are perpendicular to those of $\mathscr{R} \cap V^{\perp}$, the graph of G consists of at least two components.

Coxeter has found all graphs corresponding to the irreducible Coxeter groups. We have the following classification.

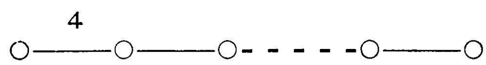
THEOREM 3.5. *Let* $\mathscr{G}$ *be a connected Coxeter graph. The following list exhausts the possibilities for* $\mathscr{G}$.

DIAGRAM 3.2

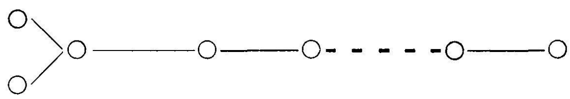
A_n $(n \geqslant 1)$



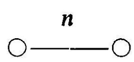
B_n $(n \geqslant 2)$



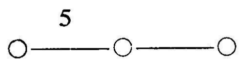
D_n $(n \geqslant 4)$



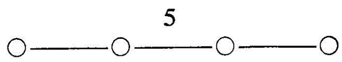
H_2^n $(n \geqslant 5)$



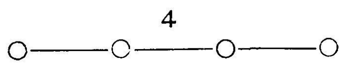
I_3



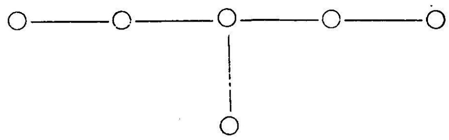
I_4



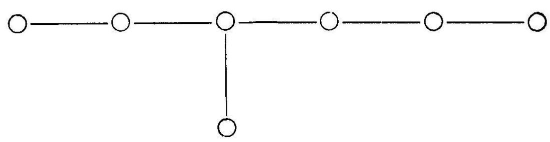
F_4



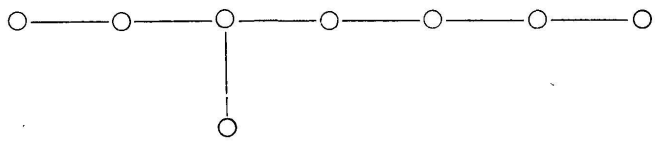
E_6



E_7



E_8



In each case the subscript denotes the number of nodes. The above list yields all irreducible Coxeter groups up to conjugacy. I.e. two irreducible groups which are conjugate subgroups of the orthogonal group have the same graph and conversely.

We give a brief description of the groups listed above.

A_n. Let S_{n+1} be the symmetric group of linear transformations $x_i' = x_{\sigma(i)}$, $1 \leqslant i \leqslant n+1$, $\sigma(i)$ being any permutation of $1, ..., n+1$. Let V $\{x \mid x_1 + ... + x_{n+1} = 0\}$ and $A_n = S_{n+1}|_V$. A_n is the group of symmetries of the regular n-simplex whose vertices are the permutations of $(-1, ..., -1, n)$.

B_n is the group of symmetries of the n cube with vertices $(\pm 1, ..., \pm 1)$. It consists of the $2^n n!$ linear transformations $x_i' = \pm x_{\sigma(i)}$, $1 \leqslant i \leqslant n$, the $\pm$ signs being chosen independently and $\sigma(i)$ an arbitrary permutation of $1, ..., n$.

D_n consists of the $2^{n-1} n!$ linear transformations $x_i' = \pm x_{\sigma(i)}$, $1 \leqslant i \leqslant n$, where $\sigma(i)$ is any permutation of $1, ..., n$ and the number of $-$ signs is even. It is readily checked that D_n is a subgroup of index 2 in B_n.

H_2^n is the dihedral group of $2n$ symmetries of the regular n-gon.

I_3 is the icosahedral group, i.e. the group of symmetries of the icosahedron.

I_4, F_4 are the groups of symmetries of certain 4-dimensional regular polytopes described in ([5], p. 156)

E_6, E_7, E_8 are the groups of symmetries of certain polytopes in R^6, R^7, R^8 known as Gosset's figures and described in ([5], p. 202)

An inspection of diagram 3.2 reveals that the graphs are of two types, those consisting of one chain and those consisting of three chains joined at a node. We refer to these graphs and their associated groups as being of types I and II. It can be shown that the groups of type I are precisely those which are the groups of symmetries of the regular polytopes ([5], p. 199).

The following theorem gives a complete description of all finite reflection groups acting on R^n.

THEOREM 3.6. *Let G be a finite reflection group acting on R^n. R^n is a direct sum of mutually orthogonal subspaces $V_0, V_1, ..., V_k$ with the following properties.*

1) *Let $G_i = G/_{V_i}$ = the restrictions of the elements of G to V_i. Then G is isomorphic to $G_0 \times G_1 \times ... \times G_k$.*
2) *G_0 consists only of the identity transformation on V_0.*

3) *Each* G_i, $1 \leqslant i \leqslant k$, *is one of the groups described in Theorem 3.5.* G *is a Coxeter group iff* $V_0 = 0$.

The proof of Theorem 3.6 is identical with that of Theorem 2.7. We simply observe that we may now choose the V_i's to be mutually orthogonal.

2. The Computation of the Degrees for Real Finite Reflection Groups

Let G be a finite irreducible orthogonal reflection group acting on the n-dimensional Euclidean space R^n. Let F be a fundamental region as described in Theorem 3.3 and $R_1, \ldots, R_n$ the n reflections in the walls of F. We shall relate the degrees $d_1, \ldots, d_n$ of the basic homogeneous invariants to the eigenvalues of $R_1 \ldots R_n$. We first prove

Theorem 3.7. *Let* $\sigma(i)$ *be any permutation of* $1, \ldots, n$. *Then* $R_1 \ldots R_n$ *is conjugate to* $R_{\sigma(1)} \ldots R_{\sigma(n)}$

Proof. Observe that $R_1 (R_1 \ldots R_n) R_1 = R_2 \ldots R_n R_1$ so that all cyclic permutations yield conjugate transformations. We may also permute any two adjacent R_i's for which the corresponding walls are orthogonal, as the R_i's then commute. Theorem 3.7 will then follow from the following

Lemma 3.1. Let $p_1, \ldots, p_n$ be nodes of a tree T. Any circular arrangement of $1, \ldots, n$ can be obtained from a sequence of interchanges of pairs i, j which are adjacent on the circle and for which p_i, p_j are not linked in T.

Proof of Lemma 3.1. We proceed by induction, the result being obvious for $n = 1$ or 2. We may assume that p_n is an end node of the tree, i.e. it links to precisely one other node. We first rearrange $1, \ldots, n-1$ as we wish. To show that this can be done, we just consider the possibility - - - *inj* - - - where p_i, p_j are not linked. If p_i, p_n are not linked, then we interchange first i, n and then i, j, obtaining - - - *nji* - - -. If p_j, p_n are not linked, then we first interchange j, n and then j, i, obtaining - - - *jin* - - -. We may therefore arrange $1, \ldots, n-1$ in the desired order. Shifting n in one direction, which is permissible as n just fails to commute with one element, we obtain the desired arrangement of $1, \ldots, n$.

In view of Theorem 3.7, the eigenvalues of $R_1 \ldots R_n$ are independent of the order in which the R_i's appear. They are also independent of the particularly chosen F. For let F' be another fundamental region as described in Theorem 3.3. Then $F' = \sigma F$, $\sigma \in G$. The reflections in the walls of F'