

3. Tabulation of the Degrees

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3. TABULATION OF THE DEGREES

Theorem 3.8 can be used to compute the degrees of the basic homogeneous invariants of G , in case G is an irreducible reflection group acting on R^n . This has been done in [7], and we tabulate these degrees below

Group	d_1, \dots, d_n
A_n ($n \geq 1$)	$2, \dots, n + 1$
B_n ($n \geq 2$)	$2, 4, \dots, 2n$
D_n ($n \geq 4$)	$2, 4, \dots, n, \dots, 2n - 4, 2n - 2$
H_2^n ($n \geq 5$)	$2, n$
E_6	$2, 5, 6, 8, 9, 12$
E_7	$2, 6, 8, 10, 12, 14, 18$
E_8	$2, 8, 12, 14, 18, 20, 24, 30$
F_4	$2, 6, 8, 12$
I_3	$2, 6, 10$
I_4	$2, 12, 20, 30$

We observe that in each case, $d_1 = 2$. This can be seen as follows. Suppose that there existed a homogeneous invariant $I(x)$ of degree 1. Since $I(\sigma x) = I(x)$ whenever $\sigma \in G$, the hyperplane $\{x \mid I(x) = 0\}$ would be a proper invariant subspace of G , contradicting that the latter is irreducible. Hence there are no homogeneous invariants of degree 1 and $d_1 \geq 2$. On the other hand, $\sum_{i=1}^n x_i^2$ is invariant under G as G is orthogonal. It follows

that $d_1 = 2$, with corresponding invariant $I_1 = \sum_{i=1}^n x_i^2$.

In applying Theorem 3.8, we must find the roots of the characteristic equation (3.23). In some cases, this is a rather tedious computation. For the groups A_n, B_n, D_n, H_2^n we can exhibit a basis of homogeneous invariants without the use of Theorem 3.8. We require

THEOREM 3.13. *Let G be a finite reflection group acting on the n -dimensional vector space V over a given field k . Let P_1, \dots, P_n be homogeneous*

invariants of G of respective degrees k_1, \dots, k_n . P_1, \dots, P_n form a basis for the invariants of $G \Leftrightarrow k_1 \dots k_n = |G|$ and

$$\Delta = \frac{\partial (P_1, \dots, P_n)}{\partial (x_1, \dots, x_n)} \neq 0.$$

Proof. By relabeling indices, we may assume $k_1 \leq \dots \leq k_n$. The \Rightarrow part of the theorem is contained in Theorems 1.2, 2.2, 2.3. Conversely, let $k_1 \dots k_n = |G|$ and $\Delta \neq 0$. Thus P_1, \dots, P_n are algebraically independent. Let I_1, \dots, I_n be basic homogeneous invariants of respective degrees d_1, \dots, d_n . Suppose $k_i = d_i, 1 \leq i \leq i_0$, but $k_{i_0+1} < d_{i_0+1}$. Then P_1, \dots, P_{i_0+1} are polynomials in I_1, \dots, I_{i_0} , implying that P_1, \dots, P_n are algebraically dependent, a contradiction. Hence $k_i \geq d_i, 1 \leq i \leq n$. Since

$$\prod_{i=1}^n d_i = \prod_{i=1}^n k_i = |G|, \text{ we must have } k_i = d_i, 1 \leq i \leq n.$$

Let $\delta_m = \dim \mathcal{J}_m, 0 \leq m < \infty, \mathcal{J}_m$ being the space of homogeneous invariants of degree m . Then $\delta_m =$ number of non-negative integral solutions to $j_1 d_1 + \dots + j_n d_n = m$. This number also equals the number of monomials $P_1^{j_1} \dots P_n^{j_n}$ which are of degree m . The algebraic independence of P_1, \dots, P_n implies that these δ_m monomials are linearly independent over k . Thus \mathcal{J}_m is spanned by these monomials for $0 \leq m < \infty$. We have shown that every homogeneous invariant is a polynomial in P_1, \dots, P_n , so that the P_i 's form a basis for the invariants of G .

We now obtain an explicit basis for the invariants of A_n, B_n, D_n, H_2^n . A_n : This group consists of the $(n+1)!$ permutations $x'_i = x_{\sigma(i)}, 1 \leq i \leq n+1$, restricted to the subspace $V = \{x \mid x_1 + \dots + x_{n+1} = 0\}$.

We choose x_1, \dots, x_n as coordinates on V . Let $P_i = \sum_{j=1}^{n+1} x_j^{i+1}, 1 \leq i \leq n$,

where $x_{n+1} = -(x_1 + \dots + x_n)$. P_i is a homogeneous invariant of degree $i+1$. We have $2 \cdot \dots \cdot (n+1) = (n+1)! = |A_n|$.

We show that $\Delta \neq 0$. Now

$$\frac{\partial P_i}{\partial x_j} = (i+1)x_j^i - (i+1)x_{n+1}^i, 1 \leq i, j \leq n.$$

Hence $\Delta = (n+1)! D$ where D is the $n \times n$ determinant whose (ij) -th entry $= x_j^i - x_{n+1}^i$. To evaluate D , we introduce the Vandermonde determinant

$$\begin{vmatrix} 1 & \dots & \dots & \dots & 1 \\ x_1 & \dots & \dots & \dots & x_{n+1} \\ x_1^n & \dots & \dots & \dots & x_{n+1}^n \end{vmatrix} = \prod_{1 \leq i < j \leq n+1} (x_j - x_i)$$

Subtracting the $(n+1)$ -th column from the first n columns, the above determinant is readily seen to equal $(-1)^n D$. Thus

$$(3.25) \quad \Delta = (-1)^{n+2} (n+1)! \prod_{1 \leq i < j \leq n+1} (x_j - x_i) = (n+1)! \prod_{1 \leq j \leq n} (x_j - x_i) \cdot \prod_{i=1}^n (x_i + s)$$

where $s = x_1 + \dots + x_n$. (3.25) shows that $\Delta \neq 0$. We conclude that $d_1 = 2, \dots, d_n = n + 1$.

B_n : Let $P_i = \sum_{j=1}^n x_j^{2i}, 1 \leq i \leq n$. P_i is a homogeneous invariant of degree $2i$. We have $2 \cdot \dots \cdot 2n = 2^n n! = |B_n|$. A computation shows that $\Delta = 2^n n! \prod_{i=1}^n x_i \prod_{1 \leq i < j \leq n} (x_j^2 - x_i^2) \neq 0$. It follows that $d_1 = 2, \dots, d_n = 2n$.

D_n : Let $P_1 = x_1 \dots x_n, P_i = \sum_{j=1}^n x_j^{2(i-1)}, 2 \leq i \leq n$. P_1 is a homogeneous invariant of degree n ; $P_i, 2 \leq i \leq n$, is a homogeneous invariant of degree $2(i-1)$. The product of the degrees = $n \cdot 2 \cdot 4 \cdot \dots \cdot (2n-2) = 2^{n-1} n! = |D_n|$.

$$(3.26) \quad \Delta = \begin{vmatrix} \frac{P_1}{x_1} & \dots & \frac{P_1}{x_n} \\ 2x_1 & \dots & 2x_n \\ \cdot & \dots & \cdot \\ 2(n-1)x_1^{2n-3} & \dots & 2(n-1)x_n^{2n-3} \end{vmatrix} = 2^{n-1} (n-1)! \prod_{1 \leq i < j \leq n} (x_j^2 - x_i^2) \neq 0$$

It follows that d_i, \dots, d_n are identical with the numbers $2, 4, \dots, n, \dots, 2n - 4, 2n - 2$.

H_2^n : Let z be the complex coordinate $x_1 + i x_2$. H_2^n may be described as the group generated by the transformation $z \rightarrow \bar{z}, z \rightarrow \zeta z$, where $\zeta = e^{\frac{2\pi i}{n}}$. Let $P_1 = x_1^2 + x_2^2, P_2 = Re z^n$. P_1, P_2 are homogeneous invariants of respective degrees $2, n$. The product of these degrees = $2n = |H_2^n|$. A computation yields

$$\frac{\partial (P_1, P_2)}{\partial (x_1, x_2)} = -2n Im z^n \neq 0.$$

It follows that $d_1 = 2, d_2 = n$.