

4. Solomon's Theorem

Objektyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **24 (1978)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **12.07.2024**

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4. SOLOMON'S THEOREM

We present in this section another method for determining the degrees of the basic invariants, valid whenever the underlying field k has characteristic 0.

THEOREM 3.14 (Solomon [18]). *Let G be a finite reflection group acting on the n -dimensional space V . Let $g_r =$ number of elements of G which fix some r -dimensional subspace of V but do not fix a subspace of higher dimension. Let d_1, \dots, d_n be the degrees of the basic homogeneous invariants of G and set $m_j = d_j - 1$. Then*

$$(3.27) \quad (t + m_1) \dots (t + m_n) = g_0 + g_1 t + \dots + g_n t^n$$

Equating the t^{n-1} -coefficients of both sides of (3.27), we obtain $g_1 = r = \sum_{i=1}^n m_i$. Setting $t = 1$ in (3.27), we obtain $\prod_{i=1}^n (m_i + 1) = \sum_{i=0}^n g_i = |G|$. Thus Theorem 3.14 generalizes Theorem 2.2.

To prove Theorem 3.14, we obtain an analog of Molien's formula for the invariant differential forms of G . We digress to a brief discussion of differential forms.

For $p > 0$, let $\omega = \sum_{i_1 < \dots < i_p} r_{i_1 \dots i_p}(x) dx_{i_1} \dots dx_{i_p}$, where $r_{i_1 \dots i_p}(x) \in k(x)$, the summation extending over all integer p -tuples satisfying $1 \leq i_1 < \dots < i_p \leq n$. ω is called a differential p -form (or simply p -form). The elements of $k(x)$ are called the 0-forms. If $\eta = \sum_{i_1 < \dots < i_p} s_{i_1 \dots i_p}(x) dx_{i_1} \dots dx_{i_p}$ is another p -form, then we define

$$\omega + \eta = \sum_{i_1 < \dots < i_p} (r_{i_1 \dots i_p} + s_{i_1 \dots i_p}) dx_{i_1} \dots dx_{i_p}.$$

Thus the p -forms constitute a vector space over $k(x)$ which we denote by \mathcal{D}_p . The elements $dx_{i_1} \dots dx_{i_p}$ form a basis for \mathcal{D}_p , so that $\dim \mathcal{D}_p = \binom{n}{p}$, $0 \leq p \leq n$. We also define a multiplication between two forms as follows. Let $dx_i dx_j = -dx_j dx_i$; in particular $dx_i dx_i = 0$. The product $\omega\eta$ of any two forms ω, η is then obtained by the distributive law. We observe that for 1-forms, $\omega\eta = -\eta\omega$, so that $\omega\omega = 0$. It follows that $\mathcal{D}_p = 0$ for $p > n$. Finally, for any rational function r , we define the 1-form dr to be

$$\sum_{i=1}^n \frac{\partial r}{\partial x_i} dx_i.$$

It is then readily checked that for n rational functions, r_1, \dots, r_n , we have

$$dr_1 \dots dr_n = \frac{\partial (r_1, \dots, r_n)}{\partial (x_1, \dots, x_n)} dx_1 \dots dx_n.$$

Let σ be a non-singular matrix with entries in k . We define

$$\sigma \omega = \sum_{i_1 < \dots < i_p} r_{i_1 \dots i_p} (\sigma^{-1}x) dx_{i_1} (\sigma^{-1}x) \dots dx_{i_p} (\sigma^{-1}x)$$

Thus σ becomes a linear transformation on each \mathcal{D}_p , interpreting the latter as a vector space over k . Let k^n be the space of n -tuples with entries in k . If G is a group of linear transformations acting on k^n , then ω is said to be invariant under G provided $\sigma\omega = \omega, \forall \sigma \in G$.

We shall prove Theorem 3.14 describing the invariant differential forms with polynomial coefficients. G is assumed throughout to be a finite reflection group acting on k^n .

LEMMA 3.4. Let I_1, \dots, I_n be basic homogeneous invariants for G . Let

$$\Pi(x) = \frac{\partial (I_1, \dots, I_n)}{\partial (x_1, \dots, x_n)}.$$

The polynomial $p(x)$ satisfies $\sigma p = (\det \sigma) p$, for every $\sigma \in G$ (in which case, we say p is skew) iff $p = \Pi i$ where i is a polynomial invariant under G .

Proof. Let $y = \sigma x$. Then

$$\begin{aligned} (3.28) \quad \Pi(x) &= \frac{\partial (I_1(y), \dots, I_n(y))}{\partial (x_1, \dots, x_n)} \\ &= \frac{\partial (I_1(y), \dots, I_n(y))}{\partial (y_1, \dots, y_n)} \det \sigma = \Pi(\sigma x) \det \sigma \end{aligned}$$

which shows that Π is skew. Hence Πi is skew for every invariant polynomial i .

Conversely, let $p(x)$ be skew. Let π be an r.h. of G with equation $L(x) = 0$. By Lemma 2.2, we may choose $v \notin \pi$, so that v is a common eigenvector to all reflections in G with r.h. π . Choose $x = Ty$, $\det T \neq 0$, so that in the y coordinates the equation of π becomes $y_n = 0$ and v becomes $(0, \dots, 0, 1)$. Let $q(y) = p(Ty)$. Let H be the subgroup of G which fixes π . By Lemma 2.2, H is a cyclic group. Let σ generate H and $h = \text{ord } H$. If ζ is the eigenvalue of σ which is a primitive h -th root of 1, then

$q(y_1, \dots, y_{n-1}, \zeta y_n) = \zeta^{-1} q(y_1, \dots, y_n)$. Writing $q = \sum q_i y_n^i$, the q_i 's being polynomials in y_1, \dots, y_{n-1} , we obtain

$$(3.29) \quad \sum q_i \zeta^{i+1} y_n^i = \sum q_i y_n^i$$

Equating coefficients in (3.29), we conclude $q_i = 0$ whenever $h \nmid i+1$. Thus $q_i = 0$ for $i < h-1 \Rightarrow y_n^{h-1} | q \Rightarrow L^{h-1} | p$. Repeating this argument for all r.h.'s of G and using Theorem 2.5, we conclude that $P = \Pi i$, where i is a polynomial. $\sigma i = \sigma P / \sigma \Pi = \frac{P}{\Pi} = i$ shows that i is invariant under G .

LEMMA 3.5. Let σ be a non-singular matrix with entries in k . Let $r \in k(x)$. Then $\sigma(dr) = d(\sigma r)$.

Proof. By definition

$$(3.30) \quad \sigma(dr) = \sum_{i=1}^n \frac{\partial r}{\partial x_i}(\sigma^{-1}x) dx_i(\sigma^{-1}x), \quad d(\sigma r) = \sum_{i=1}^n \frac{\partial}{\partial x_i}(r(\sigma^{-1}x)) dx_i$$

$$\text{Let } \sigma^{-1} = (a_{ij}). \text{ Then } x_i(\sigma^{-1}x) = \sum_{j=1}^n a_{ij} x_j \text{ and } \frac{\partial x_i}{\partial x_j}(\sigma^{-1}x) = a_{ij}.$$

Hence

$$(3.31) \quad dx_i(\sigma^{-1}x) = \sum_{j=1}^n a_{ij} dx_j$$

Applying the chain rule,

$$(3.32) \quad \frac{\partial}{\partial x_i}(r(\sigma^{-1}x)) = \sum_{j=1}^n \frac{\partial r}{\partial x_j}(\sigma^{-1}x) a_{ji}$$

Inserting (3.31), (3.32) into (3.30), we get $\sigma(dr) = d(\sigma r)$.

THEOREM 3.15. Every invariant p -form with polynomial coefficients may be expressed uniquely as

$$\sum_{i_1 < \dots < i_p} a_{i_1 \dots i_p} dI_{i_1} \dots dI_{i_p}, \quad a_{i_1 \dots i_p} \in k[I_1, \dots, I_n].$$

Proof. By Lemma 3.5, $\sigma(dI_k) = dI_k$, so that dI_1, \dots, dI_n are invariant forms. Since $\sigma(\omega\eta) = \sigma(\omega)\sigma(\eta)$ for any two forms ω, η , we conclude that

$$\sum_{i_1 < \dots < i_p} a_{i_1 \dots i_p} dI_{i_1} \dots dI_{i_p} \text{ is invariant whenever } a_{i_1 \dots i_p} \in k(I_1, \dots, I_n).$$

We show that the $\binom{n}{p}$ forms $dI_{i_1} \dots dI_{i_p}$ are linearly independent over $k(x)$, so that they form a basis for \mathcal{D}_p over $k(x)$. Suppose that

$$\sum_{i_1 < \dots < i_p} k_{i_1 \dots i_p} dI_{i_1} \dots dI_{i_p} = 0, \quad k_{i_1 \dots i_p} \in k(x).$$

Multiply this relation by $dI_{i_{p+1}} \dots dI_{i_n}$, where i_{p+1}, \dots, i_n are the indices complementary to i_1, \dots, i_p . We obtain

$$k_{i_1 \dots i_p} dI_1 \dots dI_n = k_{i_1 \dots i_p} \Pi(x) dx_1 \dots dx_n = 0 \Rightarrow k_{i_1 \dots i_p} = 0$$

for all i_1, \dots, i_p . Hence the $\binom{n}{p}$ forms $dI_{i_1} \dots dI_{i_p}$ are linearly independent over $k(x)$. It follows that every p -form ω may be expressed uniquely as

$$\omega = \sum_{i_1 < \dots < i_p} a_{i_1 \dots i_p} dI_{i_1} \dots dI_{i_p}, \quad a_{i_1 \dots i_p} \in k(x).$$

If ω is invariant, then the group averaging argument shows that $a_{i_1 \dots i_p} \in k(I_1, \dots, I_n)$. Multiply both sides of the above relation by $dI_{i_{p+1}} \dots dI_{i_n}$. We get

$$(3.33) \quad \omega dI_{i_{p+1}} \dots dI_{i_n} = \pm \Pi a_{i_1 \dots i_p} dx_1 \dots dx_n.$$

Let ω be a p -form with polynomial coefficients. We conclude from (3.33) that $\Pi a_{i_1 \dots i_p}$ is a polynomial. Since $\Pi a_{i_1 \dots i_p}$ is skew, Lemma 3.4 implies that $\Pi a_{i_1 \dots i_p} = \Pi i$, i being an invariant polynomial. Hence $a_{i_1 \dots i_p} \in k[I_1, \dots, I_n]$ for all i_1, \dots, i_p , thus proving Theorem 3.11.

THEOREM 3.16. *Let $\sigma_p(x_1, \dots, x_n)$ be the p -th elementary symmetric function in x_1, \dots, x_n (σ_0 is interpreted to be 1). Let $\omega_1(\gamma), \dots, \omega_n(\gamma)$ be the eigenvalues of γ , $\gamma \in G$. Then*

$$(3.34) \quad \frac{\sigma_p(t^{m_1}, \dots, t^{m_n})}{(1-t^{m_1+1}) \dots (1-t^{m_n+1})} = \frac{1}{|G|} \sum_{\gamma \in G} \frac{\sigma_p(\omega_1(\gamma), \dots, \omega_n(\gamma))}{(1-\omega_1(\gamma)t) \dots (1-\omega_n(\gamma)t)}, \quad 0 \leq p \leq n$$

REMARK. For $p = 0$, the above becomes formula (2.5) of Chapter II.

Proof. Let \mathcal{D}_{pm} = space of p -forms whose coefficients are homogeneous polynomials of degree m . \mathcal{D}_{pm} is a finite dimensional vector space over k . Let \mathcal{I}_{pm} = space of invariant forms in \mathcal{D}_{pm} and $d_{pm} = \dim \mathcal{I}_{pm}$. For $0 \leq p \leq n$, let $p_p(t) = \sum_{m=0}^{\infty} d_{pm} t^m$. We obtain two formulas for $p_p(t)$ by computing d_{pm} in two different ways. By Theorem 3.15, the differentials

$$I_1^{k_1} \dots I_n^{k_n} dI_{i_1} \dots dI_{i_p}, \quad m = k_1(m_1+1) \dots + k_n(m_n+1) + m_{i_1} + \dots + m_{i_p},$$

form a basis for \mathcal{J}_{pm} , so that

$$(3.35) \quad p_p(t) = \frac{\sigma_p(t^{m_1}, \dots, t^{m_n})}{(1-t^{m_1+1}) \dots (1-t^{m_n+1})}$$

Let \tilde{k} = algebraic closure of k . Define $\tilde{\mathcal{D}}_{pm}, \tilde{\mathcal{J}}_{pm}$, analogously to $\mathcal{D}_{pm}, \mathcal{J}_{pm}$, replacing k by \tilde{k} . For $\gamma \in G$, γ acts both on \mathcal{D}_{pm} and $\tilde{\mathcal{D}}_{pm}$. Let $(\text{Tr } \gamma)_{pm}$ = trace of γ as a transformation on \mathcal{D}_{pm} = trace of γ as a transformation on $\tilde{\mathcal{D}}_{pm}$. By Lemma 1.2

$$(3.36) \quad d_{pm} = \frac{1}{|G|} \sum_{\gamma \in G} (\text{Tr } \gamma)_{pm}$$

Choose T so that $T \sigma T^{-1} = D$, D being diagonal with diagonal entries $\omega_1(\gamma), \dots, \omega_n(\gamma)$. The elements $x^a dx_{i_1} \dots dx_{i_p}$, $|a| = m$ and $1 \leq i_1 < \dots < i_p \leq n$, form a basis for $\tilde{\mathcal{D}}_{pm}$. Since

$$(3.37) \quad D(x^a dx_{i_1} \dots dx_{i_p}) = [\omega(\gamma^{-1})]^a \omega_{i_1}(\gamma^{-1}) \dots \omega_{i_p}(\gamma^{-1}),$$

we have

$$(3.38) \quad (\text{Tr} D)_{pm} = \sum_{|a|=m} [\omega(\gamma^{-1})]^m \sigma_p(\omega(\gamma^{-1}))$$

(3.36), (3.38) yield

$$(3.39) \quad d_{pm} = \frac{1}{|G|} \sum_{\gamma \in G} \sum_{|a|=m} [\omega(\gamma)]^a \sigma_p[\omega(\gamma)]$$

so that

$$(3.40) \quad p_p(t) = \frac{1}{|G|} \sum_{m=0}^{\infty} \sum_{r \in G} \sum_{|a|=m} [\omega(\gamma)]^a \sigma_p(\omega(\gamma)) t^m \\ = \frac{1}{|G|} \sum_{\gamma \in G} \frac{\sigma_p(\omega(\gamma))}{(1-\omega_1(\gamma)t) \dots (1-\omega_n(\gamma)t)}$$

(3.34) follows from (3.35) and (3.40).

We derive from (3.34) the following identity.

THEOREM 3.17. For $1 \leq p \leq n$,

$$(3.41) \quad \sum_{i_1 < \dots < i_p} \frac{t^{mi_1 + \dots + mi_p}}{(1-t^{mi_1+1}) \dots (1-t^{mi_p+1})} \\ = \frac{1}{|G|} \sum_{\gamma \in G} \sum_{i_1 < \dots < i_p} \frac{\omega_{i_1}(\gamma) \dots \omega_{i_p}(\gamma)}{(1-\omega_{i_1}(\gamma)t) \dots (1-\omega_{i_p}(\gamma)t)}$$

Proof. One verifies readily, for $1 \leq p \leq n$, the identity

$$(3.42) \quad \sum_{i_1 < \dots < i_p} \frac{u_{i_1} \dots u_{i_p}}{(1 - u_{i_1} t) \dots (1 - u_{i_p} t)} \\ = \frac{h_{p_1}(t) \sigma_1(u_1, \dots, u_n) + \dots + h_{p_n}(t) \sigma_n(u_1, \dots, u_n)}{(1 - u_1 t) \dots (1 - u_n t)}$$

the u_i 's being indeterminates and the h_{p_i} 's being polynomials in t . Substitute for u_i , $\omega_i(\gamma)$ and average over the group. By Theorem 3.16, the group average becomes expression (3.42), u_i being replaced by t^{m_i} , thus proving (3.41).

We can now provide the

Proof of Theorem 3.14. Expand both sides of (3.41) in powers of $1 - t$ and equate the coefficients of $(1 - t)^{-p}$. For the left side this coefficient is

$$\sum_{i_1 < \dots < i_p} \frac{1}{(m_{i_1} + 1) \dots (m_{i_p} + 1)}$$

Let γ be an element which fixes an r dimensional subspace, but does not fix a higher dimensional subspace. This means that precisely r of the eigenvalues of γ equal 1. γ contributes to the coefficient of $(1 - t)^{-p}$ on the right side of (3.41) iff $r \geq p$, the contribution being $\binom{r}{p}$. It follows that for the right side, the $(1 - t)^{-p}$ coefficient is $\frac{1}{|G|} \sum_{r=0}^n \binom{r}{p} g_r$. Since $\prod_{i=1}^n (m_i + 1) = |G|$, we conclude that

$$(3.43) \quad \sum_{r=0}^n \binom{r}{p} g_r = \sum_{i_1 < \dots < i_{n-p}} (m_{i_1} + 1) \dots (m_{i_{n-p}} + 1), \quad 1 \leq p \leq n$$

Note that for $p = 0$, (3.43) becomes $|G| = (m_1 + 1) \dots (m_n + 1)$. Hence (3.43) also holds for $p = 0$.

The left and right side of (3.43) equal respectively $\frac{1}{p!}$ (p -th derivative at $t = 1$) of $g_0 + \dots + g_n t^n, (t + m_1) \dots (t + m_n)$. Thus $(t + m_1) \dots (t + m_n) = g_0 + \dots + g_n t^n$.