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Autor: Mac Lane, Saunders

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4. The Background in Abstract Algebra

Algebra treated by the use of axioms probably began in the late 19th century in the work of Dedekind, followed by E. H. Moore, Steinitz and others. However, modern or "abstract" algebra is a newer subject; it involves the use of axioms *and* conceptual methods to get a deeper understanding of disparate algebraic phenomena. As I have indicated elsewhere in a paper [1978] on the history of this subject, abstract algebra in this sense came into being in 1921, with a paper by Emmy Noether on "Ideal theorie in Ringbereichen". This was the first paper in which "Ring" was used in its modern axiomatic sense, though the word "number ring" had been used in Hilbert's 1898 *Zahlbericht*, while Fraenkel in 1916 had made partial attempts at axiomatics for rings. More important in this paper of Noether's was the clear recognition that some arithmetic theorems known for special rings of algebraic integers or of polynomials could be formulated and proved better under general conditions—avoiding needless computational complexities and bringing out the conceptual structures involved.

This paper of Noether's appears to have quickly stimulated many other studies in abstract algebra—both her own studies and those of her colleagues, collaborators, and pupils. In ten years, this provided the full background of abstract algebra, as formulated in van der Waerden's book *Moderne Algebra* (Band I, 1930; Band II, 1931). The abstract spirit was clearly there, though some of the central notions do not yet have due emphasis. For example, the notion of a (left) module over a ring, so important for the cohomology of groups, came in a bit indirectly under the titles "linearformenmoduln" and "groups with operators".

Related to our topic is the study of group extensions, which was stimulating by this emphasis on abstract algebra. The topic had come up before, at least implicitly, in I. Schur's study of the projective representation of a group and hence of the multiplicator. Group extension themselves were codified by Schreier [1926]. A group E with abelian normal subgroup A and quotient group G, in other words a short exact sequence

$$1 \to A \xrightarrow{i} E \xrightarrow{p} G \to 1\,, \tag{1}$$

is an *extension* of A by G. In such an extension, conjugation in E makes A a left $E/A = G$ module. If one chooses to each $x \in G$ a representative $u(x) \in E$ with $pu(x) = X$, the product of two such representatives has the form

$$u(x)\,u(y) = f(x, y)\,u(xy) \tag{2}$$

where the factor $f(x, y) \in A$ must satisfy an identity representing the associative law for the triple product $u(x)\,u(y)\,u(z)$; this f was called a "factor set". The extension E is then completely determined by G, the G-module A, and this factor set f. It turns out from the associative law that f is exactly a two dimensional cocycle for G, and that the set of all extensions E of the given G module A by G is exactly the two dimensional cohomology group $H^2(G, A)$. Hence this group, as well as the one dimension cohomology group $H^1(G, A)$, was well known in the 1930's. This made it possible for Eilenberg-Mac Lane and Eckmann to recognize in their papers cited above that the general cohomology of a group includes for dimension 2 the known case of group extensions.

In this description of group extensions by factor sets, the binary operation (of multiplication or perhaps addition) which makes $H^2(G, A)$ a group is given by the multiplication of two factor sets f, f' to form a new factor set

$$f(x, y)f'(x, y)\,.$$

In his studies of group extension [1934], Baer had raised and answered the question of finding an invariant way of describing this multiplication of two extensions (1) and (1′)—a description independent of the choice of representatives and now called the "Baer product" of extensions. He likewise had considered extensions of a *non-abelian* group A by a group G, and had observed that such an extension, realizing given operators of G on A, are not always possible. Indeed, there is a certain obstruction to forming such an extension, and this obstruction is a three-dimensional cohomology class of $H^3(G, Z)$ where Z is the center of A. This obstruction was identified in this way by Eilenberg-Mac Lane in 1947, and was a central element in the development of the cohomology of groups as an independent subject, not necessarily tied to the motivating topological examples.

5. The Background in Class Field Theory

In the early 20th century, linear algebra was an Anglo-American subject. Hamilton's discovery of quaternions and C. S. Peirce's utilization of idempotents had started the subject off. In 1905 Wedderburn had proved that any finite division algebra was commutative; one year later he proved his structure theorems. In a sense, they reduced the search for all finite