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CARTIER DUALITY AND FORMAL GROUPS OVER Z

by Joseph ROTMAN

§1. INTRODUCTION

There is an intimate relation between group theory and Lie algebra theory: the Lie algebras associated to Lie groups and to algebraic groups are powerful tools. For an abstract group, however, there is still no method of associating a Lie algebra that reveals secrets of the group. Nevertheless, when one studies abstract groups or abstract Lie algebras, he is immediately struck by analogies. It is even quite easy to construct a dictionary of such analogies containing such words as "center", "central series", "derived series", "simple"; indeed, the adjective "nilpotent" in group theory (the descending central series reaches $\{1\}$) comes from Engel's Theorem that, for such a Lie algebra, the regular representation has its image comprised of nilpotent matrices. There are also common theorems. A minor illustration: if L is a Lie algebra with center $Z(L)$, then $L/Z(L)$ is never one-dimensional; if G is a group with center $Z(G)$, then $G/Z(G)$ is never a nontrivial cyclic group. Alas, there are breakdowns: if L is a finite dimensional Lie algebra over a field of characteristic 0 and if L has trivial radical, then $L = L^2$; the false group-theoretic statement: if a finite group G has no normal solvable subgroups, then G is perfect (the symmetric group S_5 is a counter-example). Note that the ground field k of the Lie algebra was mentioned; the cited result is not true if one allows the field to have characteristic $p > 0$. Indeed, it is the aim of this paper to replace k by the ring of integers Z; one then deals with *Lie rings*, which means an additive free abelian group equipped with a multiplication satisfying the Jacobi identity and having all squares zero.

One reason for studying "formal groups" is to make precise the analogy between groups and Lie algebras. Let us give the context. The usual definition of a group G may be given with arrows. For example, multiplication is a function $m\colon G \times G \to G$; associativity asserts commutativity of the diagram

$$\begin{array}{ccc} G \times G \times G & \xrightarrow{m \times 1} & G \times G \\ {\scriptstyle 1 \times m}\downarrow & & \downarrow{\scriptstyle m} \\ G \times G & \xrightarrow{m} & G\,. \end{array}$$

The identity axiom is commutativity of the triangles

$$\begin{array}{ccccc} \{*\} \times G & \xrightarrow{\eta \times 1} & G \times G & \xleftarrow{1 \times \eta} & G \times \{*\}, \\ & \searrow & \downarrow{\scriptstyle m} & \swarrow & \\ & & G & & \end{array}$$

where $\{*\}$ is a one-point set, $\eta\colon \{*\} \to G$ is the function $* \mapsto 1 \in G$, and the slanted arrows are the obvious identifications $(*, g) \mapsto g$ and $(g, *) \mapsto g$. The reader may supply the diagram for the inverse that involves a function $i\colon G \to G$.

The point of the diagrams is that one may now define a *group-object* in a category $\mathscr{B}$ if $\mathscr{B}$ has a product $\times$ and a final object Z (to play the role of $\{*\}$). Thus, a group-object in $\mathscr{B}$ is an object B and morphisms $m\colon B \times B \to B$, $\eta\colon Z \to B$, and $i\colon B \to B$ which makes the appropriate diagrams commute. It is clear how to define homomorphisms, so that the group-objects form a subcategory $G\mathscr{B}$ of $\mathscr{B}$. Here are some easy examples: if $\mathscr{B}$ is the category of sets, then $G\mathscr{B}$ is groups; if $\mathscr{B}$ is topological spaces, then $G\mathscr{B}$ is topological groups; if $\mathscr{B}$ is groups, then $G\mathscr{B}$ is abelian groups (minor exercise). Formal groups will be group-objects in a suitable category $\mathscr{B}$.

The arrow definition of group may be dualized to define *cogroup-objects* in a category $\mathscr{A}$. If one reverses all arrows and assumes $\mathscr{A}$ has a coproduct $\amalg$ and an initial object $\aleph$, then a cogroup-object A has a comultiplication $\delta\colon A \to A \amalg A$ that is "coassociative", a "counit" $\varepsilon\colon A \to \aleph$, and a "coinverse" $j\colon A \to A$ making the appropriate dual diagrams commute. In this way, one obtains a subcategory $C\mathscr{A}$ of $\mathscr{A}$. For example, if $\mathscr{A}$ is the category of commutative k-algebras, then $C\mathscr{A}$ is the category of commutative Hopf algebras. Now Hopf algebras arise, not only as cogroup-objects in $\mathscr{A}$, but also as group-objects in another category $\mathscr{B}$. Let $\mathscr{B}$ be the category of cocommutative k-coalgebras (which, by definition, have a counit and are coassociative). An example of such a fellow is the universal enveloping algebra $U(L)$ of a Lie algebra L. It is straightforward to see that $G\mathscr{B}$ consists of cocommutative Hopf algebras, and also $U(L) \in \operatorname{obj} G\mathscr{B}$ for every Lie algebra L. This last category $G\mathscr{B}$ is essentially the formal groups we seek. It is no coincidence that Hopf algebras arose as $C\mathscr{A}$ and as $G\mathscr{B}$; with suitable hypotheses on $\mathscr{A}$ and $\mathscr{B}$, *Cartier duality* asserts these categories are antiequivalent; there are thus two ways to view formal groups.

The "good" commutative k-algebras, those corresponding to universal enveloping algebras, are rings of formal power series $k[[X_1, \dots, X_n]]$. In fact, here is the definition of formal group as it appears in [3]: a "formal group of dimension n" is a system of n formal power series $F_i(X, Y)$ in $2n$ indeterminates $X = \{X_1, \dots, X_n\}$ and $Y = \{Y_1, \dots, Y_n\}$ satisfying

$$(1) \qquad F_i(X, 0) = X \quad \text{and} \quad F_i(0, Y) = Y, \quad \text{all} \quad i;$$

$$(2) \qquad F_i(F_j(X, Y), Z) = F_i(X, F_j(Y, Z)), \quad \text{all} \quad i, j.$$

To see that this definition coincides with the definition above, just note that $k[[X_1, \dots, X_n]] \hat{\otimes} k[[X_1, \dots, X_n]] \simeq k[[X_1, \dots, X_n, Y_1, \dots, Y_n]]$, and that a comultiplication in a Hopf algebra, $\delta: k[[X]] \to k[[X]] \hat{\otimes} k[[X]]$, is completely determined by $\delta(X_i)$, $i = 1, \dots, n$. Properties (1) and (2) are the necessary constraints on δ, e.g., (2) gives coassociativity.

Before discussing Cartier duality in more detail, let us show how one links formal groups to Lie algebras. We consider $G\mathscr{B}$ as above, namely, all cocommutative Hopf algebras over a field k. If $H \in \operatorname{obj} G\mathscr{B}$ has comultiplication $\delta: H \to H \hat{\otimes} H$, then define

$$P(H) = \{x \in H : \delta x = 1 \otimes x + x \otimes 1\}.$$

It is easy to check that $P(H)$ is a k-space which is a Lie algebra under ordinary bracket $[x, y] = xy - yx$. If $\mathscr{L}$ is the category of Lie algebras over k, then $P: G\mathscr{B} \to \mathscr{L}$ is a functor. There is a functor $U: \mathscr{L} \to G\mathscr{B}$ taking $L \mapsto U(L)$, the universal enveloping algebra. These functors define an equivalence of categories when k has characteristic 0 [3, p. 49]. (In characteristic $p > 0$, these functors do not define an equivalence).

Let us return to our main topic, Cartier duality, and give its precise statement; a proof may be found in [1].

THEOREM *(Cartier Duality). Let* $\mathscr{A}$ *be the category of linearly compact commutative* k*-algebras, where* k *is a field; let* $\mathscr{B}$ *be the category of cocommutative* k*-coalgebras; for* $A \in \operatorname{obj} \mathscr{A}$, *let*

$$A^* = \operatorname{Hom}_c(A, k) = \{\text{all continuous functionals on } A\}.$$

(i) *The contravariant functor* $\mathscr{A} \to \mathscr{B}$ *given by* $A \mapsto A^*$ *is an antiequivalence of categories taking products to coproducts and final objects to initial objects.*

(ii) *The restriction of this functor is an equivalence* $(C\mathscr{A})^{op} \to G\mathscr{B}$.

Several remarks are in order. First, we shall not define "linearly compact"; its role is to guarantee that A and A^{**} are isomorphic vector spaces, and this is false for discrete infinite dimensional spaces. Second, the proof of (ii) is a routine inspection of the various diagrams, once statement (i) has been proved.

There are at least two papers giving a Cartier duality between certain categories of commutative topological k-algebras and of cocommutative k-coalgebras, where k is a commutative ring. (Ditters [2]; Morris and Pareigis [5]). We present a version of Cartier duality between certain commutative Z-algebras (= commutative rings) and cocommutative Z-coalgebras; actually, our proof works if one replaces Z by any principal ideal domain that is neither a field nor a complete discrete valuation ring. Thus, our theorem is weaker than those of Ditters and Morris-Pareigis in that the ground rings k are restricted; it is stronger than their results in that we need not assume the algebras are topological algebras. Indeed, it is easy to see our category of commutative algebras is a proper, full subcategory of the corresponding categories of Ditters and of Morris-Pareigis. We add that our proof is quite easy and all details are given.

§2. Groups

All groups are abelian and are written additively.

Definition. A subgroup A' of a group A is *cofinite* if A/A' if f.g. free (f.g. abbreviates "finitely generated").

Of course, A' cofinite implies $A = A' \oplus A''$, where $A'' \cong A/A'$.

Definition. The *cofinite topology* on a group A is that (linear) topology having a fundamental system of neighborhoods of 0 consisting of all cofinite subgroups of A.

It is clear that A is a topological group in the cofinite topology.

Suppose $A = Z^I$ for some index set I. We may also topologize A with the *product topology*, i.e., equip each factor Z with the discrete topology and consider A in the corresponding product topology. The first lemma shows that the cofinite topology gives a coordinate-free description of the product topology.

LEMMA 1. *If* $A = Z^I$ *and* I *is countable, then the cofinite topology coincides with the product topology.*

Proof. It is easy to see that, in either topology (and for any index sets I and J), every homomorphism $f\colon Z^I \to Z^J$ is sequentially continuous (if $x_n \to x$, then $f(x_n) \to f(x)$); if we assume I and J countable, then Z^I and Z^J are first countable (even metrizable), and so f is continuous.

Assume A' is cofinite in A, and A has the product topology. For finite n, we see Z^n is discrete (in either topology), whence the natural map $\pi\colon A \to A/A' \cong Z^n$ is continuous and $A' = \pi^{-1}(\{0\})$ is open.

Now assume A has the cofinite topology. If $U_i = \prod_{j\in I} X_j$, where $X_j = Z$ if $j \neq i$ and $X_j = \{0\}$ if $j = i$, then U_i is cofinite, hence open. It follows easily that every basic open set in the product topology is open in cofinite topology.

One may prove that Lemma 1 is true for any set I whose cardinal is nonmeasurable [6].

DEFINITION. The *completion* of a group A is $\varprojlim A/A'$, where A' ranges over all cofinite subgroups of A; we denote $\varprojlim A/A'$ by $A^\wedge$. There is a canonical map $\lambda\colon A \to A^\wedge$; we say A is *complete* if λ is an isomorphism.

COROLLARY 2. *If* $A = Z^I$, *where* I *is countable, then* A *is complete.*

Proof: It is easy to see that, in the product topology, A is complete in the usual metric. By Lemma 1 and [4, Theorem 13.7], the two notions of completeness coincide.

The following remarkable result of Łos is the reason we need not mention linear compactness. Let us denote $\mathrm{Hom}_Z(A, Z)$ by A^*.

LEMMA 3. *(Łos)*

(i) *Let* $A = Z^N = \prod_{n=1}^{\infty} <e_n>$. *If* $G = Z$ *or* $G = Z^{(I)}$, *the direct sum of card* I *copies of* Z, *then the map* $f \mapsto (f|<e_n>)$ *is an isomorphism* $\mathrm{Hom}_Z(A, G) \xrightarrow{\sim} \sum_{n=1}^{\infty} \mathrm{Hom}_Z(<e_n>, G)$.

(ii) *If* I *is countable, then* $(Z^I)^* \cong Z^{(I)}$.

(iii) *If* I *is countable and either* $A = Z^I$ *or* $A = Z^{(I)}$, *then* A *is reflexive in the sense that the natural map* $A \to A^{**}$ *is an isomorphism.*

Proof: [4; §94]. This Lemma is true if Z is replaced by any principal ideal domain that is neither a field nor a complete discrete valuation ring.

Again the countability assumption is too strong; one only needs the cardinal of I nonmeasurable. Also, part (i) is true for groups G other than Z and $Z^{(I)}$, namely, "slender" groups.

For any index sets I and J, there is a natural imbedding $Z^I \otimes Z^J \to Z^{I\times J}$ given by $(m_i) \otimes (n_j) \mapsto (m_i \otimes n_j)$.

LEMMA 4. *Assume I and J are countable. Then if $Z^I \otimes Z^J$ and $Z^{I\times J}$ are given the cofinite topology, then $Z^I \otimes Z^J$ is a dense subspace of $Z^{I\times J}$.*

Proof: By "subspace" we mean that the cofinite topology on $Z^I \otimes Z^J$ coincides with the relative topology $Z^I \otimes Z^J$ inherits from the larger space $Z^{I\times J}$. Let us write $A = Z^I \otimes Z^J$ and $G = Z^{I\times J}$. If G' is cofinite in G, then

$$A/G' \cap A \cong (A+G')/G' \subset G/G' ,$$

whence $G' \cap A$ is cofinite in A. Assume that A' is cofinite in A. Now A' is cofinite in A if and only if there are finitely many $f_i \in A^*$ with $A' = \cap \ker f_i$. Moreover, if $f \in A^*$ and $A' = \ker f$, then there exists a cofinite G' in G with $G' \cap A = A'$ if and only if there is $\tilde{f} \in G^*$ extending f. Thus it suffices to show we may extend $f \in (Z^I \otimes Z^J)^*$ to $\tilde{f} \in (Z^{I\times J})^*$. But this follows easily from the adjoint isomorphism and Lemma 3:

$$\begin{aligned} \mathrm{Hom}(Z^I \otimes Z^J, Z) &= \mathrm{Hom}(Z^I, \mathrm{Hom}(Z^J, Z)) \\ &= \mathrm{Hom}(Z^I, Z^{(J)}) \\ &= Z^{(I\times J)} = \mathrm{Hom}(Z^{I\times J}, Z) . \end{aligned}$$

We have shown that $Z^I \otimes Z^J$ is a subspace of $Z^{I\times J}$; it is dense because it contains the dense subgroup $Z^{(I\times J)}$.

We remark that Lemma 4 is false for some subgroups of $Z^{I\times J}$; for example, if $A = Z^{(I\times J)} \oplus \langle x \rangle$, where x has each coordinate 1, then $Z^{(I\times J)}$ is cofinite in A; the corresponding functional f on A cannot extend to $Z^{I\times J}$, for every $\tilde{f} \in (Z^{I\times J})^*$ that vanishes on $Z^{(I\times J)}$ must be 0 [4; Theorem 94.4].

LEMMA 5. *If I and J are countable, there is a natural isomorphism*

$$(Z^I \otimes Z^J)^{\wedge} \cong (Z^{(I)} \otimes Z^{(J)})^* .$$

(Recall: ^ means completion and * means dual space).

Proof: Since $Z^{(I)} \otimes Z^{(J)} \cong Z^{(I\times J)}$, the right hand side is $Z^{I\times J}$. By Lemma 4, $Z^I \otimes Z^J$ is a dense subspace of $Z^{I\times J}$, so that both have the same completion. This finishes the argument, for $Z^{I\times J}$ is complete, by Corollary 2.

COROLLARY 6. *If* I *and* J *are countable, then* $(Z^I \otimes Z^J)^\wedge \cong Z^K$, *where* K *is countable.*

Proof: Indeed, we have just seen that we may take $K = I \times J$.

LEMMA 7. *Assume* A *and* B *torsion-free. If* A' *is cofinite in* A *and* B' *is cofinite in* B, *then there is a natural isomorphism*

$$A \otimes B/(A'\otimes B + A \otimes B') \cong A/A' \otimes B/B' .$$

Proof: Define $\theta: A \otimes B \to A/A' \otimes B/B'$ by $a \otimes b \mapsto \bar{a} \otimes \bar{b}$ (where bar denotes appropriate coset); let $K = \ker \theta$. As A and B are torsion-free, they are Z-flat, and so there is a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & A \otimes B & \xrightarrow{\theta} & A/A' \otimes B/B' & \longrightarrow & 0 \\ & & \uparrow & & \uparrow = & & \vdots\uparrow & & \\ 0 & \to & A'\otimes B + A \otimes B' & \to & A \otimes B & \to & A\otimes B/(A'\otimes B + A\otimes B') & \to & 0 \end{array}$$

The dotted arrow exists and is an epimorphism, by diagram-chasing; it is an isomorphism because both right hand terms are f.g. free of the same rank (to compute the bottom quotient, observe that $A = A' \oplus A''$, $B = B' \oplus B''$, where $A'' \cong A/A'$ and $B'' \cong B/B'$).

LEMMA 8. *Let* $A = Z^I$ *and* $B = Z^J$, *where* I *and* J *are countable. The subgroups of* $A \otimes B$ *of the form* $A' \otimes B + A \otimes B'$, *where* A' *is cofinite in* A *and* B' *is cofinite in* B, *form a fundamental system of neighborhoods at* 0 *for the cofinite topology of* $A \otimes B$.

Proof: First of all, Lemma 7 shows that each of these special subgroups of $A \otimes B$ is cofinite.

Next, assume C is cofinite in $A \otimes B$, so there is an exact sequence

$$0 \longrightarrow C \longrightarrow A \otimes B \xrightarrow{\theta} F \longrightarrow 0$$

with F f.g. free. Define $A' = \{a \in A: \theta(a\otimes b) = 0 \text{ for all } b \in B\}$ and, similarly, $B' = \{b \in B: \theta(a\otimes b) = 0 \text{ for all } a \in A\}$. Clearly $A' \otimes B + A \otimes B' \subset C$. Now A' is pure in A and B' is pure in B, so that A/A' and B/B' are torsion-free. Also, A' is closed in A (and B' is closed in B)

because θ is continuous (I and J are countable), so that A/A' is complete. By considering maximal independent subsets of A and B and observing that only finitely many elements of A are involved in lifting a (finite) basis of F, we see that A/A' has finite rank (similarly for B/B'). As the only finite rank complete groups are f.g. free, it follows that A' and B' are cofinite.

§3. Formal Groups

Definition. Let $\mathscr{A}$ denote the category of all commutative rings with 1 whose underlying additive group is of the form Z^I, where card $I \leqslant \aleph_0$.

Note that $Z[[x_1, ..., x_n]]$, formal power series over Z in n variables, is an object of $\mathscr{A}$. Further, $\mathscr{A}$ has an initial object, namely, Z.

Lemma 9. *Every* $A \in \text{obj}\, \mathscr{A}$ *is a complete topological ring in the cofinite topology.*

Proof: By Lemma 1 and Corollary 2, we know A is a complete topological group. It remains to show that multiplication $m\colon A \times A \to A$ is continuous, and, for this it suffices to prove the corresponding homomorphism $m'\colon A \otimes A \to A$ is continuous; this is so because every homomorphism is continuous in the cofinite topology.

The next lemma is taken almost verbatim from [1; p. 12].

Lemma 10. *If* $A \in \text{obj}\, \mathscr{A}$, *then* A *has a fundamental system of neighborhoods of* 0 *consisting of cofinite ideals.*

Proof: Let A' be a cofinite subgroup of A. Since multiplication is continuous, there is a cofinite subgroup W of A with $W^2 \subset A'$. Since W is cofinite, it has a f.g. free complement $<a_1, ..., a_r>$. For each j, the continuity of $x \mapsto a_j \cdot x$ at 0 implies the existence of a cofinite $W_j \subset W$ with $a_j W_j \subset A'$. If $U = \bigcap_{j=1}^{r} W_j$, then U is cofinite in A. Moreover, $a_j U \subset A'$ for all j and $WU \subset A'$ (in fact, $W^2 \subset A'$ and $U \subset W$); hence $AU \subset A'$. Since $1 \in A$, we have $U \subset AU$, so that A/AU is f.g. Now if $(AU)_*$ is the pure subgroup of A generated by AU, then $(AU)_*$ is also an ideal, is cofinite, and $(AU)_* \subset A'_* = A'$ (for A' is already pure).

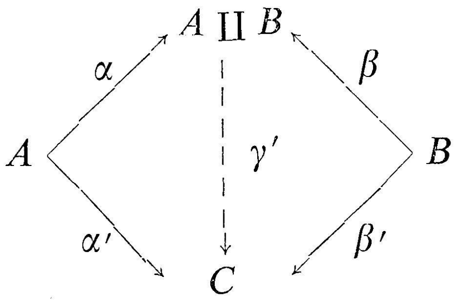
Lemma 11. *$\mathscr{A}$ has coproducts.*

Proof: If $A, B \in \text{obj}\,\mathscr{A}$, define $A \amalg B = (A \otimes B)^{\wedge}$. Observe that $A \amalg B$ has the correct additive structure, by Corollary 6. By Lemmas 7 and 8,

$$(A \otimes B)^{\wedge} \cong \varprojlim A \otimes B/(A' \otimes B + A \otimes B') = \varprojlim (A/A' \otimes B/B'),$$

where A' and B' are cofinite subgroups. By Lemma 10, we may assume A' and B' are cofinite ideals. It follows that $A \amalg B$ is a commutative ring with 1, i.e., $A \amalg B \in \text{obj}\,\mathscr{A}$.

To see that we have a coproduct, consider the diagram



where $\alpha\colon a \mapsto a \otimes 1$, $\beta\colon b \mapsto 1 \otimes b$, $C \in \text{obj}\,\mathscr{A}$, and α', β' are ring maps. Since im α and im β lie in $A \otimes B \subset A \amalg B$, the fact that $A \otimes B$ is a coproduct in the category of commutative rings with 1 provides a unique ring map $\gamma\colon A \otimes B \to C$ with $\gamma\alpha = \alpha'$ and $\gamma\beta = \beta'$. As C is complete, however, γ has a unique extension $\gamma'\colon A \amalg B \to C$ making the diagram above commute.

DEFINITION. Let $\mathscr{B}$ be the category of cocommutative Z-coalgebras whose underlying additive group is of the form $Z^{(I)}$, where card $I \leqslant \aleph_0$. (N.B. All coalgebras are, by definition, coassociative and have a counit.)

If L is a f.g. Lie ring (i.e., a Lie ring whose additive group is f.g. free), then its universal enveloping algebra is an object of $\mathscr{B}$. Note also that $\mathscr{B}$ has a final object, namely, Z.

PROPOSITION 12. *There is an antiequivalence of categories* $\mathscr{A}^{op} \xrightarrow{\sim} \mathscr{B}$ *given by* $A \mapsto A^* = \text{Hom}_Z(A, Z)$ *taking products to coproducts and final objects to initial objects.*

Proof: By Lemma 3, we know that $A^{**} = A$ (and, if $B \in \text{obj}\,\mathscr{B}$, then $B^{**} = B$). It remains to consider multiplication $m\colon A \otimes A \to A$. As A is complete, we may regard $m\colon A \amalg A \to A$. Write $A = B^*$ qua groups. Then Lemma 5 gives

$$A \amalg A = B^* \amalg B^* = (B^* \otimes B^*)^{\wedge} = (B \otimes B)^*,$$

whence multiplication may be viewed as a map $m\colon (B\otimes B)^* \to B^*$. Thus $m^*\colon B \to B \otimes B$. This comultiplication is coassociative and cocommutative (because m is associative and commutative). Finally, the unit $u\colon Z \to A = B^*$ yields a counit $u^*\colon B \to Z$. Thus $B = A^* \in \text{obj } \mathscr{B}$.

The rest of the argument follows as in [1; Chapter I, §13]; we merely give notation and results.

DEFINITION. Let $G\mathscr{B}$ denote the category of all group-objects in $\mathscr{B}$ (call such objects *formal groups over* Z); let $C\mathscr{A}$ denote the category of all cogroup-objects in $\mathscr{A}$.

LEMMA 13. *$A \in \text{obj } C\mathscr{A}$ if and only if A is a commutative Hopf algebra with $A \in \text{obj } \mathscr{A}$; $B \in \text{obj } G\mathscr{B}$ if and only if B is a cocommutative Hopf algebra with $B \in \text{obj } \mathscr{B}$.*

N.B. (By Hopf algebra, we mean a Z-bialgebra with antipode.)

We may now state our version of Cartier duality.

THEOREM 14. *There is an equivalence of categories $(C\mathscr{A})^{op} \xrightarrow{\sim} G\mathscr{B}$ implemented by $A \mapsto A^* = \text{Hom}_Z(A, Z)$.*

Proof: Precisely as in [1], using Proposition 12.

Let us now compare our result with that of Morris and Pareigis [5]. For a commutative ring k, they consider a category $k\text{-Alg}_{pf}$ defined as a certain full subcategory of all commutative topological k-algebras. When $k = Z$, this is their analogue of our category $\mathscr{A}$. In essence, a commutative topological ring A (=Z-algebra) lies in $Z\text{-Alg}_{pf}$ if $A \cong \varprojlim D_i$, where $\{D_i, p_i^j\}$ is an inverse system with directed index set of discrete commutative rings D_i that are f.g. free as abelian groups and the p_i^j are ring surjections. There is further hypothesis on the inverse system, but suffice it to say that our Z-algebras in $\mathscr{A}$ do lie in $Z\text{-Alg}_{pf}$; moreover, continuity of every ring map in $\mathscr{A}$ shows that $\mathscr{A}$ is a full subcategory of $Z\text{-Alg}_{pf}$. Since $Z\text{-Alg}_{pf}$ may contain algebras of cardinal larger than continuum, $\mathscr{A}$ is genuinely smaller than $Z\text{-Alg}_{pf}$.

In [2], Ditters gives a Cartier duality in which the analogue of $\mathscr{A}$ is called Al_Z: its objects are all commutative topological Z-algebras that are isomorphic to Z^I as a Z-module for some index set I (not necessarily countable) and such that the topology on Z^I is the product topology (each Z being discrete).

THEOREM 15. *The category* $\mathscr{A}$ *is a proper, full subcategory of the category* Z-Alg_{pf} *of Morris-Pareigis; the category* $\mathscr{A}$ *is a proper, full subcategory of the category* Al_Z *of Ditters.*

REFERENCES

[1] DIEUDONNÉ, J. *Introduction to the Theory of Formal Groups.* Marcel Dekker, New York 1973.

[2] DITTERS, E. J. Curves and formal *(co)* groups. *Inven. Math. 17* (1972), pp. 1-20.

[3] FRÖHLICH, A. *Formal Groups.* Lecture Notes in Mathematics, No. 74, Springer, Berlin, 1968.

[4] FUCHS, L. *Infinite Abelian Groups.* Academic Press, New York, volume I, 1970; volume II, 1973.

[5] MORRIS, R. A. and B. PAREIGIS. Formal groups and Hopf algebras over discrete rings. *Trans. Amer. Math. Soc. 197* (1974), pp. 113-129.

[6] NUNKE, R. J. On direct products of infinite cyclic groups. *Proc. Amer. Math. Soc. 13* (1962), pp. 66-71.

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