§1. Introduction

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CARTIER DUALITY AND FORMAL GROUPS OVER Z

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§1. INTRODUCTION

There is an intimate relation between group theory and Lie algebra theory: the Lie algebras associated to Lie groups and to algebraic groups are powerful tools. For an abstract group, however, there is still no method of associating a Lie algebra that reveals secrets of the group. Nevertheless, when one studies abstract groups or abstract Lie algebras, he is immediately struck by analogies. It is even quite easy to construct a dictionary of such analogies containing such words as "center", "central series", "derived series", "simple"; indeed, the adjective "nilpotent" in group theory (the descending central series reaches $\{1\}$) comes from Engel's Theorem that, for such a Lie algebra, the regular representation has its image comprised of nilpotent matrices. There are also common theorems. A minor illustration: if L is a Lie algebra with center $Z(L)$, then $L/Z(L)$ is never one-dimensional; if G is a group with center $Z(G)$, then $G/Z(G)$ is never a nontrivial cyclic group. Alas, there are breakdowns: if L is a finite dimensional Lie algebra over a field of characteristic 0 and if L has trivial radical, then $L = L^2$; the false group-theoretic statement: if a finite group G has no normal solvable subgroups, then G is perfect (the symmetric group S_5 is a counter-example). Note that the ground field k of the Lie algebra was mentioned; the cited result is not true if one allows the field to have characteristic $p > 0$. Indeed, it is the aim of this paper to replace k by the ring of integers Z; one then deals with *Lie rings*, which means an additive free abelian group equipped with a multiplication satisfying the Jacobi identity and having all squares zero.

One reason for studying "formal groups" is to make precise the analogy between groups and Lie algebras. Let us give the context. The usual definition of a group G may be given with arrows. For example, multiplication is a function m: $G \times G \to G$; associativity asserts commutativity of the diagram

$$\begin{array}{ccc} G \times G \times G & \xrightarrow{m \times 1} & G \times G \\ {\scriptstyle 1 \times m}\big\downarrow & & \big\downarrow{\scriptstyle m} \\ G \times G & \xrightarrow{m} & G\,. \end{array}$$

The identity axiom is commutativity of the triangles

$$\begin{array}{ccccc} \{*\} \times G & \xrightarrow{\eta \times 1} & G \times G & \xleftarrow{1 \times \eta} & G \times \{*\}, \\ & \searrow & \big\downarrow{\scriptstyle m} & \swarrow & \\ & & G & & \end{array}$$

where $\{*\}$ is a one-point set, $\eta: \{*\} \to G$ is the function $* \mapsto 1 \in G$, and the slanted arrows are the obvious identifications $(*, g) \mapsto g$ and $(g, *) \mapsto g$. The reader may supply the diagram for the inverse that involves a function $i: G \to G$.

The point of the diagrams is that one may now define a *group-object* in a category $\mathscr{B}$ if $\mathscr{B}$ has a product $\times$ and a final object Z (to play the role of $\{*\}$). Thus, a group-object in $\mathscr{B}$ is an object B and morphisms $m: B \times B \to B$, $\eta: Z \to B$, and $i: B \to B$ which makes the appropriate diagrams commute. It is clear how to define homomorphisms, so that the group-objects form a subcategory $G\mathscr{B}$ of $\mathscr{B}$. Here are some easy examples: if $\mathscr{B}$ is the category of sets, then $G\mathscr{B}$ is groups; if $\mathscr{B}$ is topological spaces, then $G\mathscr{B}$ is topological groups; if $\mathscr{B}$ is groups, then $G\mathscr{B}$ is abelian groups (minor exercise). Formal groups will be group-objects in a suitable category $\mathscr{B}$.

The arrow definition of group may be dualized to define *cogroup-objects* in a category $\mathscr{A}$. If one reverses all arrows and assumes $\mathscr{A}$ has a coproduct $\amalg$ and an initial object $\aleph$, then a cogroup-object A has a comultiplication $\delta: A \to A \amalg A$ that is "coassociative", a "counit" $\varepsilon: A \to \aleph$, and a "coinverse" $j: A \to A$ making the appropriate dual diagrams commute. In this way, one obtains a subcategory $C\mathscr{A}$ of $\mathscr{A}$. For example, if $\mathscr{A}$ is the category of commutative k-algebras, then $C\mathscr{A}$ is the category of commutative Hopf algebras. Now Hopf algebras arise, not only as cogroup-objects in $\mathscr{A}$, but also as group-objects in another category $\mathscr{B}$. Let $\mathscr{B}$ be the category of cocommutative k-coalgebras (which, by definition, have a counit and are coassociative). An example of such a fellow is the universal enveloping algebra $U(L)$ of a Lie algebra L. It is straightforward to see that $G\mathscr{B}$ consists of cocommutative Hopf algebras, and also $U(L) \in \operatorname{obj} G\mathscr{B}$ for every Lie algebra L. This last category $G\mathscr{B}$ is essentially the formal groups we seek. It is no coincidence that Hopf algebras arose as $C\mathscr{A}$ and as $G\mathscr{B}$; with suitable hypotheses on $\mathscr{A}$ and $\mathscr{B}$, *Cartier duality* asserts these categories are antiequivalent; there are thus two ways to view formal groups.

The "good" commutative k-algebras, those corresponding to universal enveloping algebras, are rings of formal power series $k[[X_1, \ldots, X_n]]$. In fact, here is the definition of formal group as it appears in [3]: a "formal group of dimension n" is a system of n formal power series $F_i(X, Y)$ in $2n$ indeterminates $X = \{X_1, \ldots, X_n\}$ and $Y = \{Y_1, \ldots, Y_n\}$ satisfying

$$(1) \qquad F_i(X, 0) = X \quad \text{and} \quad F_i(0, Y) = Y, \quad \text{all} \quad i;$$

$$(2) \qquad F_i(F_j(X, Y), Z) = F_i(X, F_j(Y, Z)), \quad \text{all} \quad i, j.$$

To see that this definition coincides with the definition above, just note that $k[[X_1, \ldots, X_n]] \hat{\otimes} k[[X_1, \ldots, X_n]] \simeq k[[X_1, \ldots, X_n, Y_1, \ldots, Y_n]]$, and that a comultiplication in a Hopf algebra, $\delta: k[[X]] \to k[[X]] \hat{\otimes} k[[X]]$, is completely determined by $\delta(X_i)$, $i = 1, \ldots, n$. Properties (1) and (2) are the necessary constraints on δ, e.g., (2) gives coassociativity.

Before discussing Cartier duality in more detail, let us show how one links formal groups to Lie algebras. We consider $G\mathscr{B}$ as above, namely, all cocommutative Hopf algebras over a field k. If $H \in \operatorname{obj} G\mathscr{B}$ has comultiplication $\delta: H \to H \hat{\otimes} H$, then define

$$P(H) = \{x \in H : \delta x = 1 \otimes x + x \otimes 1\}.$$

It is easy to check that $P(H)$ is a k-space which is a Lie algebra under ordinary bracket $[x, y] = xy - yx$. If $\mathscr{L}$ is the category of Lie algebras over k, then $P: G\mathscr{B} \to \mathscr{L}$ is a functor. There is a functor $U: \mathscr{L} \to G\mathscr{B}$ taking $L \mapsto U(L)$, the universal enveloping algebra. These functors define an equivalence of categories when k has characteristic 0 [3, p. 49]. (In characteristic $p > 0$, these functors do not define an equivalence).

Let us return to our main topic, Cartier duality, and give its precise statement; a proof may be found in [1].

THEOREM *(Cartier Duality). Let* $\mathscr{A}$ *be the category of linearly compact commutative k-algebras, where k is a field; let* $\mathscr{B}$ *be the category of cocommutative k-coalgebras; for* $A \in \operatorname{obj} \mathscr{A}$, *let*

$$A^* = \operatorname{Hom}_c(A, k) = \{\text{all continuous functionals on } A\}.$$

(i) *The contravariant functor* $\mathscr{A} \to \mathscr{B}$ *given by* $A \mapsto A^*$ *is an antiequivalence of categories taking products to coproducts and final objects to initial objects.*

(ii) *The restriction of this functor is an equivalence* $(C\mathscr{A})^{op} \to G\mathscr{B}$.

Several remarks are in order. First, we shall not define "linearly compact"; its role is to guarantee that A and A^{**} are isomorphic vector spaces, and this is false for discrete infinite dimensional spaces. Second, the proof of (ii) is a routine inspection of the various diagrams, once statement (i) has been proved.

There are at least two papers giving a Cartier duality between certain categories of commutative topological k-algebras and of cocommutative k-coalgebras, where k is a commutative ring. (Ditters [2]; Morris and Pareigis [5]). We present a version of Cartier duality between certain commutative Z-algebras (= commutative rings) and cocommutative Z-coalgebras; actually, our proof works if one replaces Z by any principal ideal domain that is neither a field nor a complete discrete valuation ring. Thus, our theorem is weaker than those of Ditters and Morris-Pareigis in that the ground rings k are restricted; it is stronger than their results in that we need not assume the algebras are topological algebras. Indeed, it is easy to see our category of commutative algebras is a proper, full subcategory of the corresponding categories of Ditters and of Morris-Pareigis. We add that our proof is quite easy and all details are given.

§2. Groups

All groups are abelian and are written additively.

Definition. A subgroup A' of a group A is *cofinite* if A/A' if f.g. free (f.g. abbreviates "finitely generated").

Of course, A' cofinite implies $A = A' \oplus A''$, where $A'' \cong A/A'$.

Definition. The *cofinite topology* on a group A is that (linear) topology having a fundamental system of neighborhoods of 0 consisting of all cofinite subgroups of A.

It is clear that A is a topological group in the cofinite topology.

Suppose $A = Z^I$ for some index set I. We may also topologize A with the *product topology*, i.e., equip each factor Z with the discrete topology and consider A in the corresponding product topology. The first lemma shows that the cofinite topology gives a coordinate-free description of the product topology.