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Artikel: SIMPLE PROOF OF THE MAIN THEOREM OF ELIMINATION THEORY IN ALGEBRAIC GEOMETRY

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- (i) As a ring, A is generated by $A_0 \cup A_1$.
- (ii) For any nonnegative integer d , A_d is a finitely generated module over A_0 .

Furthermore, let \mathfrak{S} be the ideal in A_0 consisting of all a 's such that $aA_d = 0$ for all sufficiently large d 's, i.e. the union of the annihilators of the A_0 -modules A_0, A_1, A_2, \dots .

THEOREM D. *Let $A = \bigoplus_{d \geq 0} A_d$ be a graded commutative ring obeying hypotheses (i) and (ii) above. Let K be an algebraically closed field and $\varphi : A_0 \rightarrow K$ be a ring homomorphism. In order that φ extend to a ring homomorphism $\Psi : A \rightarrow K$ which does not annihilate the ideal $A^+ = \bigoplus_{d \geq 1} A_d$ in A , it is necessary and sufficient that φ annihilate the ideal \mathfrak{S} defined above.*

We leave to the reader the simple proof of the necessity in theorem D as well as the derivation of theorem C from theorem D.

4. PROOF OF THEOREM D

Let \mathfrak{P} be the kernel of φ , a prime ideal in A_0 . Assume $\mathfrak{S} \subset \mathfrak{P}$. We subject the ring A to a number of transformations. At each step, the properties (i) and (ii) enunciated before the statement of theorem D will be preserved, as well as property $A_d \neq 0$ for every $d \geq 0$. We shall mention what has been achieved after each step.

a) Factor A through the following graded ideal J : an element a in A belongs to J if and only if there exists an element s in A_0 such $s \notin \mathfrak{P}$ and $sa = 0$. For every $d \geq 0$, the annihilator \mathfrak{S}_d of the A_0 -module A_d is contained in \mathfrak{S} hence in \mathfrak{P} and this implies $J \cap A_d \neq A_d$. Put $A' = A/J$, $\mathfrak{P}' = (\mathfrak{P} + J)/J$ and $\Sigma = A'_0 - \mathfrak{P}'$. Then any element in Σ is regular in A' .

b) Enlarge A' by replacing it by the subring A'' of the total quotient ring of A' consisting of the fractions with denominators in Σ . Let A''_d be the set of fractions with numerator in A'_d and denominator in Σ ; then $A'' = \bigoplus_{d \geq 0} A''_d$. Then A''_0 is a local ring with maximal ideal $\mathfrak{P}'' = \mathfrak{P}' \cdot A'_0$.

c) Factor A'' through the graded ideal $\mathfrak{P}'' \cdot A''$. Since A''_d is a finitely generated module over the local ring A''_0 , one gets $A''_d \neq \mathfrak{P}'' A''_d$ by Nakayama's lemma. Put $k = A''_0 / \mathfrak{P}''$, and $R = A'' / \mathfrak{P}'' A''$.

At this point, k is a field (the quotient field of A_0/\mathfrak{P}) and R is a graded algebra over the field k , so all assumptions of theorem B are fulfilled. Moreover let ε the composition of the natural maps

$$A \rightarrow A' \rightarrow A'' \rightarrow R.$$

In degree 0, ε_0 is nothing else than the natural map from A_0 into k with kernel \mathfrak{P} . Since φ has the same kernel \mathfrak{P} , it factors through ε_0 , making K an algebraically closed extension of k .

We quote now theorem B. There exists a k -linear ring homomorphism $f: R \rightarrow K$ such that $f(R^+) \neq 0$. The composite map $\Psi = f\varepsilon$ has all the required properties.

5. APPLICATION TO SCHEMES

We keep the notation of theorem D. Recall that the spectrum $S = \mathbf{Spec}(A_0)$ of A_0 is the set of all prime ideals in A_0 ; the projective spectrum $X = \mathbf{Proj}(A)$ of A is the set of all *graded* prime ideals in A , which do not contain the ideal $A^+ = \bigoplus_{d \geq 1} A_d$. We have a natural map $\pi: X \rightarrow S$ associating to every graded prime ideal \mathfrak{P} in A the prime ideal $\mathfrak{P} \cap A_0$ in A_0 .

Moreover S and X are endowed with their respective Zariski topologies. A set F in S (resp. X) is closed if and only if there exists an ideal \mathfrak{A} in A_0 (resp. A) such that F is the set of ideals \mathfrak{P} of S (resp. X) containing \mathfrak{A} . It is obvious that π is continuous.

The following theorem is Grothendieck's version of the elimination theorem. Using his language, it is the main step in the proof that $X = \mathbf{Proj}(A)$ is a proper scheme over $S = \mathbf{Spec}(A_0)$.

THEOREM E. *The map $\pi: X \rightarrow S$ is closed, that is the image of a closed set is closed.*

Let $F \subset X$ be closed and let \mathfrak{A} be an ideal in A such that F consists of the graded prime ideals \mathfrak{P} of X containing \mathfrak{A} . Replacing if necessary \mathfrak{A} by the ideal generated by the homogeneous components of its elements, we may and shall assume that \mathfrak{A} is a graded ideal. Let \mathfrak{B} be the set of elements a in A_0 such that $a \cdot A_d \subset \mathfrak{A}$ for large d , and let G be the set of prime ideals in A_0 containing \mathfrak{B} . It is obvious that π maps F into G .

Let \mathfrak{P}_0 be a prime ideal in G , hence $\mathfrak{P}_0 \supset \mathfrak{A}_0$ (where $\mathfrak{A}_0 = \mathfrak{A} \cap A_0$). Denote by k the quotient field of A_0/\mathfrak{P}_0 and by K an algebraically closed