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Artikel: SIMPLE PROOF OF THE MAIN THEOREM OF ELIMINATION THEORY IN ALGEBRAIC GEOMETRY

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At this point, k is a field (the quotient field of A_0/\mathfrak{P}) and R is a graded algebra over the field k , so all assumptions of theorem B are fulfilled. Moreover let ε the composition of the natural maps

$$A \rightarrow A' \rightarrow A'' \rightarrow R.$$

In degree 0, ε_0 is nothing else than the natural map from A_0 into k with kernel \mathfrak{P} . Since φ has the same kernel \mathfrak{P} , it factors through ε_0 , making K an algebraically closed extension of k .

We quote now theorem B. There exists a k -linear ring homomorphism $f: R \rightarrow K$ such that $f(R^+) \neq 0$. The composite map $\Psi = f\varepsilon$ has all the required properties.

5. APPLICATION TO SCHEMES

We keep the notation of theorem D. Recall that the spectrum $S = \mathbf{Spec}(A_0)$ of A_0 is the set of all prime ideals in A_0 ; the projective spectrum $X = \mathbf{Proj}(A)$ of A is the set of all *graded* prime ideals in A , which do not contain the ideal $A^+ = \bigoplus_{d \geq 1} A_d$. We have a natural map $\pi: X \rightarrow S$ associating to every graded prime ideal \mathfrak{P} in A the prime ideal $\mathfrak{P} \cap A_0$ in A_0 .

Moreover S and X are endowed with their respective Zariski topologies. A set F in S (resp. X) is closed if and only if there exists an ideal \mathfrak{A} in A_0 (resp. A) such that F is the set of ideals \mathfrak{P} of S (resp. X) containing \mathfrak{A} . It is obvious that π is continuous.

The following theorem is Grothendieck's version of the elimination theorem. Using his language, it is the main step in the proof that $X = \mathbf{Proj}(A)$ is a proper scheme over $S = \mathbf{Spec}(A_0)$.

THEOREM E. *The map $\pi: X \rightarrow S$ is closed, that is the image of a closed set is closed.*

Let $F \subset X$ be closed and let \mathfrak{A} be an ideal in A such that F consists of the graded prime ideals \mathfrak{P} of X containing \mathfrak{A} . Replacing if necessary \mathfrak{A} by the ideal generated by the homogeneous components of its elements, we may and shall assume that \mathfrak{A} is a graded ideal. Let \mathfrak{B} be the set of elements a in A_0 such that $a \cdot A_d \subset \mathfrak{A}$ for large d , and let G be the set of prime ideals in A_0 containing \mathfrak{B} . It is obvious that π maps F into G .

Let \mathfrak{P}_0 be a prime ideal in G , hence $\mathfrak{P}_0 \supset \mathfrak{A}_0$ (where $\mathfrak{A}_0 = \mathfrak{A} \cap A_0$). Denote by k the quotient field of A_0/\mathfrak{P}_0 and by K an algebraically closed

overfield of k . Let φ be the natural composite map $A_0/\mathfrak{A}_0 \rightarrow A_0/\mathfrak{P}_0 \rightarrow k \rightarrow K$. We are now in a position to apply theorem D to the graded ring A/\mathfrak{A} , and we get a ring homomorphism $\Psi : A/\mathfrak{A} \rightarrow K$ extending φ and such that $\Psi((A^+ + \mathfrak{A})/\mathfrak{A}) \neq 0$. Let \mathfrak{P}_d (for $d \geq 1$) be the set of elements a in A_d such that $\Psi(a + \mathfrak{A}) = 0$. Then $\mathfrak{P} = \bigoplus_{d \geq 0} \mathfrak{P}_d$ is a graded prime ideal in A containing \mathfrak{A} with $\mathfrak{P} \not\supset A^+$ and $\mathfrak{P} \cap A_0 = \mathfrak{P}_0$. That is, \mathfrak{P} belongs to F and π maps \mathfrak{P} onto \mathfrak{P}_0 .

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