6. Betti Numbers or Homology Groups

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Others (Eilenberg-Mac Lane and Eckmann) dutifully cited Teichmüller's results—but it seems unlikely that those results really affected the development.

6. Betti Numbers or Homology Groups

The period 1927-1937 saw an extensive algebraization of combinatorial topology; this process was an essential prerequisite to the cohomology of groups. Before 1927, topology really was combinatorial: a chain in a complex was a string of simplices, each perhaps affected with a multiplicity (a coefficient), and the algebraic manipulation of chains was something auxiliary to their geometric meaning. This is undoubtedly as it must be, at the start; only later can it develop that geometric results follow from long algebraic computations which are not geometrically visible, step by step.

Combinatorial topology, following Poincaré, measured the connectivity of a polyhedron by its Betti numbers and torsion coefficients in each dimension, calculated as they were from chains and their boundaries. Between 1927 and 1934, the style changed completely; now the connectivity was measured by the homology groups, one in each dimension; the invariants of these abelian groups gave the previous Betti numbers and torsion coefficients. It is fascinating to trace this change, as best we now can. I can find no mention of homology *groups* before 1927; for example, the famous 1915 and 1926 papers of Alexander proved the invariance of the Betti *numbers* of a complex, not the invariance of the homology groups. Veblen's *Analysis Situs* (first edition, 1921; second edition, 1931) is all phrased in terms of incidence matrices and Betti numbers, except for one brief section in the back of the book where it is noted that the homology classes module p form a group.

Then in 1927 Vietoris studied the homology of spaces which were not necessarily polyhedra, so that the homotopy groups were not necessarily finitely generated—so of course he (had to) use homology groups. W. Mayer, with references to courses by Vietoris, used homology groups in a 1929 paper on "Abstract Topology" (submitted, November 1927). Heinz Hopf reviewed the paper in the *Jahrbuch*. In his review he notes, evidently with some surprise, that the paper used "group-theoretic methods". E. R. van Kampen's Dutch thesis "Die Combinatorische Topologie und die Dualitits-satz", Den Haag 1929, formulates these ideas by homology *groups*. An influential article by Van der Waerden in 1930 summarized the state of topology: he used homology groups. Alexandroff (whose 1928 papers about compacta were all in terms of Betti numbers) used homology groups in his 1932 monograph on topology—but Alexandroff's review in the Jahrbuch of the 1927 paper by Vietoris doesn't even notice the use of groups.

A folk tale has it that homology groups first appeared in Göttingen. In the period 1926-1932 A. D. Alexandroff and Heinz Hopf frequently visited there; I heard Alexandroff lecture there on topology in 1932. At one time, perhaps in 1926, they were studying with some difficulty Lefschetz's proof of his fixed point theorem. They discussed it with Emmy Noether, who pointed out that the proof could be better understood by replacing the Betti numbers with the corresponding homology groups and using the trace of a suitable endomorphism of these groups. Other versions of the folk tale have it that Emmy simply observed that Betti numbers and torsion coefficients should be viewed as the standard invariants of a suitable abelian group, which should be the proper tool for the conceptual formulation of homological connectivity. It is not now clear whether or not this was the first use of the homology *groups*. At any rate, it is the case that these groups appear as such in the small 1932 book in which Alexandroff recorded his Göttingen lectures, while the 1935 book of Alexandroff and Hopf gives credit in the Preface to the advice of Emmy Noether.

In this case, it is difficult to identify a first use. It seems most likely that many topologists independently came to use homology groups rather than Betti numbers—and that this easy transition, much in keeping with the growth of abstract algebra, was not noted in any way as a special event. Only after the fact do we note a change—the development of mathematics in hindsight is seen under a very different perspective than at the time.

The use of homology groups was but a small part of the algebraization of topology. Another vital step clearly related to our story was the introduction of cohomology groups and the (cup product) cohomology ring. Previously one had used intersections for chains on manifolds. In 1935 it appeared that these intersections could be dualized to cup products of cochains—and that in this form the products would hold not just for manifolds, but for any polyhedra. This situation was recognized independently by Alexander, by Čech, Kolmogoroff and by Whitney; all in reports which they (except for Čech) planned to give at the 1935 conference on topology in Moscow. Whitney ultimately replaced his talk by one on another topic, but formulated his results on cup products in his decisive 1938 paper.

These observations about groups and homology may suffice to understand the trend 1927-1938 toward a thoroughgoing algebraic formulation of homology.