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## 7. The Background in Homotopy

Group theory, in fact had been present in combinatorial topology from the beginning, in the study of the fundamental group (the Poincaré group) of a space or in particular of a manifold. The fundamental group $\Pi_1$ for a polyhedron $P$ naturally comes with a presentation of the form $\Pi_1 = F/R$, where $F$ is a suitable free group generated by circuits in the one-skeleton of $P$, while its subgroup $R$ is described from the 2-cells of $P$. Hence *this* sort of presentation was ready at hand for Hopf's study of the influence of the fundamental group—and his paper does make reference to the work of Reidemeister, one of the German topologists concerned with the fundamental group.

The introduction of the higher homotopy group was more recent. At the 1932 International Congress of Mathematicians in Zurich, E. Čech had described our present two-dimensional homotopy group in a very brief note. He wrote no further on the subject. Folklore has it that other topologists at the conference discouraged him from further work, pointing out that his $\Pi_2$ was an abelian group, while all the experience with $\Pi_1$ indicated that what was wanted was a non-abelian group. Hence the real credit for the higher homotopy groups goes to W. Hurewicz, who introduced them in several brief notes in 1935-36, together with proofs of several of their properties—enough to show that these higher homotopy groups *did* have utility in topology. In particular, his 1936 theorem *that* the homology groups of an aspherical polyhedron are determined by the fundamental group of that polyhedron is the exact starting point of our subject.

Other developments at this time emphasized the importance of homotopy —Hopf's discovery [1931] of the essential maps of $S^3$ on $S^2$, and the work of Whitehead on combinatorial homotopy. It was clearly the right time to investigate the relation between homotopy and homology.

## 8. The Cohomology of Groups

Once launched by topology, the higher dimensional cohomology groups of a group took on a life of their own. Eilenberg-Mac Lane and Mac Lane separately examined properties of the group $H^n(G, A)$ for a general $G$-module $A$. They found (from the study of Baer) the purely group-theoretical interpretation of $H^3(G, A)$ by obstructions—but an equally useful interpretation for higher $n$, long sought for, is still missing (and may even not be there!). Eckmann introduced $G$-finite cohomology groups (1947) and showed their connection with the Hopf-Freudenthal theory of the *ends* of a group. Eckmann's work, and the paper of Eilenberg-Mac Lane on complexes with operators, again emphasized the connection of cohomology groups of groups with covering spaces. There was a systematic presentation of the subject in the Cartan seminar of 1950/51, entitled "Cohomologie des groupes, suite spectrale, faisceaux". In this seminar Eilenberg first described the cohomology groups axiomatically, and then proved their existence. Subsequent exposés by Cartan emphasized the calculation of the cohomology by free resolutions complete with an abstract version of the comparison theorem. A decisive example of the effective use of such resolutions is the calculation of the cohomology of a cyclic group—carried out here in exposé 3. (I am sensitive to the advantage of using resolutions for this purpose, because in 1948 I had calculated the cohomology of cyclic groups directly from the bar resolution *without* the general comparison theorem—the direct method worked but was much more cumbersome.) Subsequent exposés made a number of applications—to the Brauer group, the Wedderburn theorem, the theorem of Maschke on complete reducibility of linear representations of a finite group, and P. A. Smith's theorem.

Further applications to pure group theory have been limited. One small but striking one is the homology proof by Gaschutz [1966]:

THEOREM. A finite non-abelian $p$-group has an automorphism of $p^{\text{th}}$ power order which is not an inner automorphism.

This conference in Zurich has exhibited more examples of the use of homology in group theory.

## 9. SPECTRAL SEQUENCES

The results stimulated by group cohomology were not confined just to group theory. For example, the problem of computing the cohomology groups $H^n(G, A)$ for the case when $G$ itself is a group extension (say, cyclic by cyclic) immediately leads to the study of a spectral sequence. Specifically, if

$$1 \to K \to G \to Q \to 1 \qquad (1)$$

is a short exact sequence of (multiplicative) groups and $A$ is a left $G$ module there is a spectral sequence $E_r^{pq}$ with