§ 2. The Lefschetz trace formula for the c.f.p. index

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This reduces to the classical Lefschetz-Hopf theorem if B = a point, or if $p: E = B$. Our assumptions in 2.1 are a little more restrictive than necessary, in order to facilitate the proof; a slight generalization is indicated in 2.8.

(2.1) Theorem. *Let* $p: E \to B$ *be an* ENR_B, *where* B *is a* compact *ENR. Let* $g: D_g \to E$, $\varphi: D_\varphi \to B$ *denote maps as in 1.1 such that* Fix (g) is compact, and $D_\varphi \supset$ Fix (g). *Then the c.f.p. index of* (g, φ) *agrees with the Lefschetz trace of the composite* $\check{h}B \xrightarrow{\varphi} \check{h}$ Fix$(g) \xrightarrow{t} \check{h}B$, *or* $hB \xrightarrow{\varphi^*} hD \xrightarrow{t^D} hB$, *where* $t = t_g$ *is the fixed-point transfer* (cf. [2], § 3), D *is any neighborhood of* Fix (g) *in* D_φ, h *is singular and* $\check{h}$ *is* $\check{C}$*ech-cohomology with coefficients in* $\mathbf{Z}$ *or* $\mathbf{Q}$. In formulas,

$$(2.2) \qquad J(g, \varphi) = tr(t_g \circ \check{\varphi}) = tr(t_g^D \circ \varphi^*).$$

Proof. Using a vertical neighborhood retraction we can assume that $E = \mathbf{R}^n \times B$; this is, in fact, what the definition 1.6-1.9 shows (if $Y = \mathbf{R}^n$). Then $g(y, b) = (\gamma(y, b), b)$, where $\gamma: D_g \to \mathbf{R}^n$, and $J(g, \varphi) = I(\gamma, \varphi)$ as explained in 1.12. Furthermore, since B is ENR, we have $\iota: B \subset U \subset \mathbf{R}^m$ and a retraction $\rho: U \to B$, where U is open in $\mathbf{R}^m$. We can then extend φ, γ, g to maps $\tilde{\varphi}, \tilde{\gamma}, \tilde{g}$ of open subsets of $\mathbf{R}^n \times U \subset \mathbf{R}^n \times \mathbf{R}^m$ by composing with $id \times \rho: \mathbf{R}^n \times U \to \mathbf{R}^n \times B$. The fixed points of (γ, φ), $(\tilde{\gamma}, \tilde{\varphi})$ (and their index) are the same, by commutativity [1], VIII, 5.16 — since $(\tilde{\gamma}, \tilde{\varphi}) = (id \times \iota)(\gamma, \varphi)(id \times \rho)$. Altogether (omitting the $\tilde{}$), we can assume that φ, γ, g are defined in open subsets D_φ, $D_\gamma = D_g$ of $\mathbf{R}^n \times U$, $\varphi: D_\varphi \to B \subset U$, $\gamma: D_g \to \mathbf{R}^n$, $D_\varphi \supset$ Fix (g), Fix (g) is (no longer compact but) proper over U; in particular, K = Fix $(g) \cap (p^{-1}B)$ is compact.

We now argue in a similar (although simpler) fashion as on p. 241 of [2]. We consider the following diagram (explanations below).

(2.3)

$$
\begin{array}{ccccccc}
\mathbf{R}^n_o \times \mathbf{R}^m_o & \overset{\alpha}{\dashrightarrow} & (X, X-K) & & \xrightarrow{(q\text{-}\gamma,\ p\text{-}\varphi)} & & \mathbf{R}^n_o \times \mathbf{R}^m_o \\
\Big\downarrow{\scriptstyle id \times j} & & \Big\| & & & & \Big\uparrow{\scriptstyle id \times d} \\
\mathbf{R}^n_o \times (U, U-B) & \overset{\beta}{\dashrightarrow} & \big(X, (X-\mathrm{Fix}(g)) \cup (X-X_B)\big) & \xrightarrow{(q\text{-}\gamma,\ id)} & \mathbf{R}^n_o \times (X, X-X_B) & \xrightarrow{id \times (p,\, \varphi)} & \mathbf{R}^n_o \times (U, U-B) \times B
\end{array}
$$

$$\mathbf{R}^n_o \times (U, U-B) \xrightarrow{\ t_g\ } \mathbf{R}^n_o \times (X, X-X_B)$$

Here, $\mathbf{R}^n_o = (\mathbf{R}^n, \mathbf{R}^n - 0)$, X is an open neighborhood of $\operatorname{Fix}(g)$ in which γ and φ are defined, $K = \operatorname{Fix}(g) \cap (p^{-1}B)$, $X_B = X \cap (p^{-1}B)$, $q\colon X \subset \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^n$ is the projection, $d(u, b) = u - b$. The dotted arrows stand for sequences of inclusion maps (as in [2], 3.3); some of these go the wrong way but then they are homotopy equivalences or excisions, inducing isomorphisms in cohomology. For instance, α stands for

$$\mathbf{R}^n_o \times \mathbf{R}^m_o = \mathbf{R}^{n+m}_o \sim (\mathbf{R}^{n+m}, \mathbf{R}^{n+m} - C) \hookrightarrow (\mathbf{R}^{n+m}, \mathbf{R}^{n+m} - K) \xleftarrow{EXC} \supset (X, X\text{-}K)$$

where C is a ball around 0, containing K. Similarly for j on the left. β is a relative version (compare [2], 3.7), namely

$$\begin{aligned}
&\mathbf{R}^n_o \times (U, U - B) \sim (\mathbf{R}^n, \mathbf{R}^n - C') \times (U, U - B) \hookrightarrow \\
&\big(\mathbf{R}^n \times U,\ (\mathbf{R}^n \times U - \operatorname{Fix}(g)) \cup (\mathbf{R}^n \times (U - B)\big) \xleftarrow{EXC} \supset \\
&\big(X, (X - \operatorname{Fix}(g)) \cup (X - X_B)\big),
\end{aligned}$$

where C' is a ball around $0 \in \mathbf{R}^n$ such that $K \subset (C' \times B)$. The lower t_g will be explained later.

The reader might want to follow the track of an element across the diagram 2.3; it looks as follows

$$\begin{array}{ccccccc}
(y, b) & \mapsto & (y, b) & \longmapsto & & & (y - \gamma(y, b),\ b - \varphi(y, b)) \\
\downarrow & & \| & & & & \uparrow \\
(y, b) & \mapsto & (y, b) & \longmapsto & (y - \gamma(y, b),\ y, b) & \longmapsto & (y - \gamma(y, b), b, \varphi(y, b)).
\end{array}$$

We now apply cohomology $h = H^*(-; \mathbf{Q})$ to the diagram 2.3. Let $s^n \in h^n \mathbf{R}^n_o$ the canonical generator. Then $s^n \times s^m$ generates $h^{n+m}(\mathbf{R}^n_o \times \mathbf{R}^m_o)$, and its image along the top row of 2.3 is $I(g, \gamma)\, s^n \times s^m = J(g, \varphi)\, s^n \times s^m$, by definitions [1], VII, 5.2, and 1.9 above.

The left part of the lower row (which is marked t_g) induces the relative transfer (or trace) homomorphism $t_g\colon h(X, X - X_B) \to h(U, U - B)$, as defined in [2], 3.6-8. In formulas,

(2.4) $$s^n \times \xi \quad s^n \mapsto \times\ t_g^{X, Z}(\xi),\quad Z = X - X_B .$$

Actually, [2], 3.8 is a little more general: it maps $h(X, X - X_B)$ into $h(U, \tilde{U})$, where $\tilde{U} \supset (U - B)$; we've composed [2], 3.8 with $h(U, \tilde{U}) \to h(U, U - B)$.

Using the Künneth-formula we can write

(2.5) $$d^*s^m = \sum_\nu \alpha_\nu \times \beta_\nu\,, \quad \text{with } \alpha_\nu \in h(U, U-B)\,,\ \beta_\nu \in hB\,.$$

Following $\alpha_\nu \times \beta_\nu$ along the lower row of (2.3) gives

(2.6) $$\alpha_\nu \times \beta_\nu \mapsto t_g(p^*\alpha_\nu \smile \varphi^*\beta_\nu) = \alpha_\nu \smile (t_g\varphi^*\beta_\nu)\,,$$

the latter because t_g is a homomorphism of modules over $h(U, U-B)$, by the relative version of [2], 3.20.

If we define $\kappa\colon h(U, U-B) \to \mathbf{Q}$ by $j^*(u) = \kappa(u)\, s^m$ (this corresponds to γ on p. 233, line 3^- of [2]), then $s^n \times \alpha_\nu \times \beta_\nu$ has image $\kappa(\alpha_\nu \smile t_g\varphi^*\beta_\nu)\, s^n \times s^m$ in the upper left corner of 2.3. On the other hand $\kappa(\alpha_\nu \smile t_g\varphi^*\beta_\nu)$ is the trace of the endomorphism

$$\xi \mapsto (-1)^{|\beta_\nu|}\beta_\nu\, \kappa(\alpha_\nu \smile t_g\varphi^*\xi)\,, \quad \xi \in hB\,,$$

by [2], 6.7. It follows, that the image of $d^*s^m = \sum_\nu s^n \times \alpha_\nu \times \beta_\nu$ in the upper left corner is $s^n \times s^m$-times the trace of

(2.7) $$\xi \mapsto \sum_\nu (-1)^{|\beta_\nu|}\beta_\nu\, \kappa(\alpha_\nu \smile t_g\varphi^*\xi)\,, \quad \xi \in hB\,,$$

and so $J(g, \varphi) =$ trace of 2.7.

It remains to show that 2.7 agrees with $t_g^B\varphi_B^*$, where we now add indices (B, or U) to indicate the range of t_g resp. the domaine of φ^*. This will follow from [2], 6.16 which asserts (in greater generality) that $\sum_\nu (-1)^{|\beta_\nu|}\beta_\nu\, \kappa(\alpha_\nu \smile \eta) = \iota^*\eta$, for $\eta \in hU$ and $\iota^*\colon hU \to hB$. Taking $\eta = t_g^U\varphi_U^*\xi$ we see that 2.7 agrees with $\xi \mapsto \iota^* t_U^g\varphi_U^*\xi = t_g^B\varphi_B^*\xi$, the latter by naturality ([2], 3.12) of t_g applied to ι. □

(2.8) *Remark.* The assumption in 2.1 that B be compact can be weakened: It suffices that for some compact subset $R \subset B$ we have that $\mathrm{Fix}(g)_R = \mathrm{Fix}(g) \cap (p^{-1}R)$ *is compact, and*

$$\mathrm{im}(\varphi) \subset R\,, \quad D_\varphi \supset \mathrm{Fix}(g)_R\,.$$

Then the composite $\check{h}R \xrightarrow{\check{\varphi}} \check{h}(\mathrm{Fix}(g)_R) \xrightarrow{t_g} \check{h}R$ *is defined, has finite rank, and has Lefschetz trace equal to* $J(g, \varphi)$.

Our proof of 2.1 can be adapted to this more general situation. *Or*, by arguments as in [2], 8.6, one can slightly increase R in B, and decrease D_φ, such that the increased R is a compact ENR, and over (the increased) R the assumptions of 2.1 are satisfied; then 2.1 will imply the more general result above.

§ 3. Applications, Problems.

(3.1) Whether and how the trace formula 2.1 can be used depends mainly on one's knowledge of the transfer t_g. For instance, one knows that

(i) $t_g p^* = I(g_b) =$ multiplication with the Hopf-index of $g_b \colon D_g \cap p^{-1}b \to p^{-1}b$ (in ordinary cohomology, B connected).

(ii) $t_g \colon hD_g \to hB$ is induced by a stable map of B^+ into D_g^+; in particular, it commutes with stable cohomology operations.

(iii) t_g is itself given by a trace-formula if $p \colon E \to B$ is a bundle with compact fibres which are totally non-cohomologous to zero.

We shall now illustrate (cf. 3.2, 3.3, 3.5) how these properties can be used.

(3.2) *Suppose* φ *is homotopic to* $\beta(p \mid D_\varphi)$, *for some* $\beta \colon B \to B$. Then $t_g \varphi^* = t_g p^* \beta^* = I(g_b)\beta^*$, provided B is connected (cf. [2], 4.8). Therefore

$$J(g, \varphi) = tr(t_g \varphi^*) = I(g_b)\, tr(\beta^*) = I(g_b)\, I(\beta)\,.$$

Geometrically, this result is very plausible: If $\varphi = \beta_p$ then Coinc (φ, p) consists of all fibres $D_\varphi \cap p^{-1}b$ with $b \in \mathrm{Fix}(\beta)$. The "number" of these fibres is $I(\beta)$, and in every fibre the "number" of fixed points of g equals $I(g_b)$. — As the geometry suggests, the result holds under more general assumptions and can be proved directly from § 1 (it doesn't seriously use 2.1).

As an illustration, the reader might look at the case where $p \colon E \to B$ is the tangent sphere-bundle of a compact Riemannian manifold B, and $\varphi = \varphi_t \colon E \to B$, $\varphi(x) = \exp(tx)$, for $t \in \mathbf{R}$. Clearly $\varphi \simeq \varphi_0 = p$, and Coinc $(\varphi, p) = \varnothing$ if $|t|$ is small enough, $t \neq 0$. Hence, $0 = J(g, \varphi) = I(g_b)\, I(id_B) = I(g_b)\, \chi(B)$, for all g. (For a direct proof of this result the reader should think of $\mathrm{Fix}(g) \subset E$ as a manifold such that $p \mid \mathrm{Fix}(g)$ has degree $I(g_b)$).

(3.3) The definition [2], 3.3-4 shows that t_g is a composite of geometric homomorphisms (induced by continuous maps) and suspension isomor-