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Our proof of 2.1 can be adapted to this more general situation. *Or*, by arguments as in [2], 8.6, one can slightly increase R in B, and decrease D_φ, such that the increased R is a compact ENR, and over (the increased) R the assumptions of 2.1 are satisfied; then 2.1 will imply the more general result above.

§ 3. Applications, Problems.

(3.1) Whether and how the trace formula 2.1 can be used depends mainly on one's knowledge of the transfer t_g. For instance, one knows that

(i) $t_g p^* = I(g_b) =$ multiplication with the Hopf-index of $g_b\colon D_g \cap p^{-1}b \to p^{-1}b$ (in ordinary cohomology, B connected).

(ii) $t_g\colon hD_g \to hB$ is induced by a stable map of B^+ into D_g^+; in particular, it commutes with stable cohomology operations.

(iii) t_g is itself given by a trace-formula if $p\colon E \to B$ is a bundle with compact fibres which are totally non-cohomologous to zero.

We shall now illustrate (cf. 3.2, 3.3, 3.5) how these properties can be used.

(3.2) *Suppose* φ *is homotopic to* $\beta(p \mid D_\varphi)$, *for some* $\beta\colon B \to B$. Then $t_g\varphi^* = t_g p^* \beta^* = I(g_b)\,\beta^*$, provided B is connected (cf. [2], 4.8). Therefore

$$J(g, \varphi) = tr(t_g\varphi^*) = I(g_b)\, tr(\beta^*) = I(g_b)\, I(\beta)\,.$$

Geometrically, this result is very plausible: If $\varphi = \beta_p$ then Coinc (φ, p) consists of all fibres $D_\varphi \cap p^{-1}b$ with $b \in \mathrm{Fix}(\beta)$. The "number" of these fibres is $I(\beta)$, and in every fibre the "number" of fixed points of g equals $I(g_b)$. — As the geometry suggests, the result holds under more general assumptions and can be proved directly from § 1 (it doesn't seriously use 2.1).

As an illustration, the reader might look at the case where $p\colon E \to B$ is the tangent sphere-bundle of a compact Riemannian manifold B, and $\varphi = \varphi_t\colon E \to B$, $\varphi(x) = \exp(tx)$, for $t \in \mathbf{R}$. Clearly $\varphi \simeq \varphi_0 = p$, and Coinc $(\varphi, p) = \varnothing$ if $|t|$ is small enough, $t \neq 0$. Hence, $0 = J(g, \varphi) = I(g_b)\, I(id_B) = I(g_b)\,\chi(B)$, for all g. (For a direct proof of this result the reader should think of $\mathrm{Fix}(g) \subset E$ as a manifold such that $p \mid \mathrm{Fix}(g)$ has degree $I(g_b)$).

(3.3) The definition [2], 3.3-4 shows that t_g is a composite of geometric homomorphisms (induced by continuous maps) and suspension isomorphisms $h^j Y \cong h^{j+n}((\mathbf{R}^n, \mathbf{R}^n - 0) \times Y)$. Thus, t_g is induced by a stable map $B^+ \to D_g^+$ (in fact, by a stable shape map $B^+ \to \mathrm{Fix}(g)^+$); it commutes with stable cohomology operations, such as Steenrod's Sq^i or P^i. As a (rather weak) consequence of theorem 2.1 we obtain that *under the assumptions of 2.1 the c.f.p.-index* $J(g, \varphi)$ *is the Lefschetz trace of a homomorphism* $hB \to hB$ *which is induced by a stable map* $B^+ \to B^+$; *this homomorphism (namely* $t_g \varphi^*$*) satisfies* $1 \mapsto I(g_b) \cdot 1$.

For example, let $B = P_{2m}\mathbf{C}$ = complex projective $2m$-space, hence $H^*(B; R) = R[u]/(u^{2m+1})$, with $u \in H^2$, R any ring. If $R = \mathbf{Z}/2\mathbf{Z}$ then $Sq^2 u^{2i-1} = u^{2i}$; a stable map α must therefore satisfy $\alpha^* u^j = \lambda_j u^j$ with $\lambda_{2i-1} = \lambda_{2i}$. For integral coefficients $R = \mathbf{Z}$ this means $\lambda_{2i-1} \equiv \lambda_{2i} \bmod 2$. Therefore $tr(\alpha^*) \equiv \lambda_0 \bmod 2$. In our case $\alpha = t_g \varphi^*$ this says:

Under the assumptions of 2.1 and with $B = P_{2m}\mathbf{C}$ *the c.f.p.-index satisfies* $J(g, \varphi) \equiv I(g_b) \bmod 2$. In particular, *if* $I(g_b)$ *is odd then* $J(g, \varphi) \neq 0$, *hence every* φ *has coincidence points with* p.

It is interesting to compare this result with [3], where the product case $E = Y \times P_{2m}\mathbf{C}$ is treated by different methods. It is shown there (compare also 1.12) that $J(g, \varphi)$, for globally defined (g, φ), is equal to $I(g_b)$ times an odd integer; in particular, $I(g_b) \neq 0 \Rightarrow J(g, \varphi) \neq 0$. One might wonder whether this extends to general bundles over $P_{2m}\mathbf{C}$, but the following example shows that it doesn't. Let $B = P_{2m}\mathbf{C}$, $E = B \times B - \tilde{\Delta}$ where $\tilde{\Delta}$ is an open tubular neighborhood of the diagonal, φ and p the two projections onto B, $g = id_E$. Then p is a bundle projection with compact fibre $\simeq P_{2m-1}\mathbf{C}$, $I(g_b) = \chi(\text{fibre}) = 2m \neq 0$, but $\mathrm{Coinc}(\varphi, p) = \varnothing$.

(3.4) If $p: E \to B$ is a fibration (where E and B are compact ENR, B connected) and if the fibre $Y = p^{-1}(b)$ is totally non-cohomologous to zero, i.e. $hE \to hY$ is epimorphic for $h = H^*(-; \mathbf{Q})$, then E is h-flat over B in the sense of [2], 6.9; in fact, hE has a Leray-Hirsch basis ([2], 6.8) over hB. In particular, $hE \cong hY \otimes hB$, as hB-modules (but not as rings, in general). In this case, [2], 6.18 expresses t_g in terms of Lefschetz traces over the ring hB. One can combine the two trace-formulas 2.1 and [2], 6.18, as follows.

(3.5) Proposition. *Let* $p: E \to B$ *a fibration between compact ENR-spaces* E, B *(*B *connected), and let* $\iota: Y \hookrightarrow E$ *the inclusion of the fibre. Assume* $hE = hY \otimes hB$ *as* hB*-modules, and such that* $\iota^*(y \otimes 1) = y$ *for* $y \in hY$, *where* $h = H^*(-; \mathbf{Q})$. *Then for every map* $\varphi: E \to B$ *and fibre-*

preserving map $g: E \to E\,(pg = p)$ *the c.f.p.-index* $J(g, \varphi)$ *equals the Lefschetz trace of*

$$hY \otimes hB \to hY \otimes hB\,, \; y \otimes z \mapsto g^*(y \otimes 1) \smile (\varphi^* z)\,.$$

Heuristically, this is found by pretending that the isomorphism $hE = hY \otimes hB$ comes from a product representation, and by comparing 2.1 with the discussion 1.12 of the product case. In order to actually prove it, we consider the following purely algebraic construction. For every $\alpha \in \mathrm{Hom}_{hB}(hE, hE) = \mathrm{Hom}_{\mathbf{Q}}(hY, hE)$ we define $\tau_\alpha \in \mathrm{Hom}_{hB}(hE, hB)$ by $\tau_\alpha(\xi) = tr\,(\tilde{\xi} \circ \alpha)$, where $\xi \in hE$ and $\tilde{\xi} \in \mathrm{Hom}_{hB}(hE, hE)$ is left translation with ξ, $\tilde{\xi}(x) = \xi \smile x$. For $\beta \in \mathrm{Hom}_{\mathbf{Q}}(hB, hE)$ and α as above, we define $\{\alpha, \beta\} \in \mathrm{Hom}_{\mathbf{Q}}(hE, hE)$ by $\{\alpha, \beta\}(y \otimes z) = \alpha(y \otimes 1)(\beta z)$. We assert,

(3.6) $$tr\,\{\alpha, \beta\} = tr\,(\tau_\alpha \circ \beta)\,.$$

If we take $\alpha = g^*$ then $\tau_\alpha = t_g$, by [2], 6.18. If, moreover, $\beta = \varphi^*$ then 3.6 becomes 3.5, by 2.1. Thus, it remains to give a

Proof of 3.6. Let $\{y_i\}$ resp. $\{z_j\}$ denote bases of $hY = H^*(Y; \mathbf{Q})$ resp. $hB = H^*(B; \mathbf{Q})$. Since both sides of 3.6 are bilinear in (α, β) it suffices to consider the case where α and β vanish on all but one basic element y_i resp. z_j; thus, $\alpha(y_\mu) = 0$ for $\mu \neq i$, $\beta(z_\nu) = 0$ for $\nu \neq j$. Then

$$\{\alpha, \beta\}(y_i \otimes z_j) = (\alpha y_i) \smile (\beta z_j) = \lambda(y_i \otimes z_j) + \rho\,,$$

where $\lambda \in \mathbf{Q}$, and the remainder term ρ is irrelevant for the trace; hence, $tr\,\{\alpha, \beta\} = (-1)^{|y_i| + |z_j|}\lambda$, where $|\;|$ denotes dimension. Similarly, $(\widetilde{(\beta z_j)} \circ \alpha)(y_i) = (\beta z_j) \smile (\alpha y_i) = (-1)^{|y_i||z_j|}\, y_i \otimes (\lambda z_j) + \rho$, hence $(\tau_\alpha \circ \beta)(z_j) = tr\,(\widetilde{(\beta z_j)} \circ \alpha) = (-1)^{|y_i|}\lambda z_j + \rho'$ by [2], 6.6, hence $tr\,(\tau_\alpha \circ \beta) = (-1)^{|z_j|}(-1)^{|y_i|}\lambda$. □

(3.7) *Multivalued maps* $\beta: B \to B$ are usually given by, resp. related to pairs of ordinary maps $B \xleftarrow{p} E \xrightarrow{\varphi} B$ such that $\beta(x) = \varphi p^{-1}(x)$ resp. $\beta(x) \supset \varphi p^{-1}(x)$. Fixed points of β can then be obtained from coincidence points of (φ, p) since $\mathrm{Fix}(\beta) \supset p\,(\mathrm{Coinc}(\varphi, p))$. The existence theorems in the literature (cf. [4], and its informative bibliography) often assume that p is a Vietoris-map (i.e. proper, with acyclic fibres). Then $p^*: hB \to hE$ is isomorphic in Čech-cohomology h, and the Lefschetz trace of $(p^*)^{-1}\varphi^*: hB$

$\to hB$ can be used to detect fixed points of β. This is clearly related to our theorem 2.1. It appears less general than 2.1 because 2.1 makes no acyclicity-assumption (but if $p: E \to B$ is Vietoris and $D_\varphi = D_g = E$ then $t_g = (p^*)^{-1}$). On the other hand, it has a more general aspect than 2.1 because it doesn't assume an actual fibration (or ENR_B), only a "cohomology fibration" (with "pointlike" fibres). This comparison suggests a common generalization, namely to general *cohomology fibrations* $p: E \to B$ with suitable compactness and ANR-properties. The main step for such a program would be to construct transfer homomorphisms $t_g: hE \to hB$ for proper (co-)homology fibrations. This is an interesting problem in itself but may involve a fair amount of technicalities; for some applications in coincidence theory it could perhaps be bypassed by directly generalizing 3.5 to cohomology fibrations.

(3.8) *Remarks.* If one is primarily interested in coincidence points of $B \xleftarrow{p} E \xrightarrow{\varphi} B$ the methods of this paper can be of help but they are not entirely adequate, not even when generalized as suggested in 3.7. The point is that they are not going after Coinc (φ, p) itself, but rather after the intersection of Coinc (φ, p) with Fix (g). It should be possible to measure Coinc (φ, p) itself, in terms of (co-)homology invariants. If B is manifold then one can use $(\varphi, p)^* (\tau)$, where τ is the Thom-class of the diagonal of $B \times B$. For products $E = Y \times B$, or fibrations as in 3.5, one can define an invariant κ in $\oplus_j (H^j Y \otimes H_j B)$. It seems plausible that this can be adapted to rather general $B \xleftarrow{p} E \xrightarrow{\varphi} B$, at least if B is ENR. But one would expect the invariant to be hard to compute — harder than $J(\varphi, p)$ anyway.

Instead of intersecting Coinc (φ, p) with sets of the form Fix (g) one could probably mimic this process on a (co-)homology level and intersect with other classes than those of the form $\{ \text{Fix}(g) \}$. For instance, in the product case it would presumably amount to taking scalar-products of κ with elements in $H_j Y \otimes H^j B$. Again, one would expect that these numbers are harder to deal with than $J(\varphi, g)$. On the other hand, it seems quite possible that the traces $\Lambda(h)$ in [5], or those of [6] could be obtained in this way.