ON A FUNCTIONAL EQUATION RELATING TO THE BRAUER-RADEMACHER IDENTITY

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Let f be a completely multiplicative arithmetical function. That is, f is a complex valued function defined on the set of positive integers such that

$$f(mn) = f(m)f(n)$$

for all m and n. We assume throughout that f is not identically zero, so that we have $f(1) = 1$. Also, let h be a multiplicative function, that is,

$$h(mn) = h(m)\,h(n) \quad \text{whenever} \quad (m, n) = 1\,,$$

where (m, n) denotes the greatest common divisor of m and n. The Dirichlet convolution of f and h is defined by

$$F(n) = \sum_{d \mid n} f(d)\,h\left(\frac{n}{d}\right). \tag{1}$$

In 1960, E. Cohen (cf. [6], Theorem 3.2) established that the functional equation

$$F(n) \sum_{\substack{d \mid n \\ (m,d)=1}} \frac{f(d)}{F(d)}\,\mu\left(\frac{n}{d}\right) = \mu(n) \sum_{d \mid (m,n)} f(d)\,h\left(\frac{n}{d}\right) \tag{2}$$

is satisfied when $h(n) = \mu(n)\,g(n)$, where g is a multiplicative function and μ is the Möbius function, under the following assumptions:

$$f(p) \neq 0 \text{ and } f(p) - g(p) \neq 0 \text{ for all primes } p. \tag{3}$$

These assumptions on f and g are made to insure the non vanishing of $F(n)$ for all n (cf. [4], Lemma 2), so that the quotient under $\sum$ in (2) is meaningful.

A famous special case of (2), which was established by A. Brauer and H. Rademacher [1] in 1926 is the following (here $f(n) = n$, $h(n) = \mu(n)$):

$$\phi(n) \sum_{\substack{d \mid n \\ (m,d)=1}} \frac{d}{\phi(d)}\,\mu\left(\frac{n}{d}\right) = \mu(n) \sum_{d \mid (m,n)} d\mu\left(\frac{n}{d}\right), \tag{4}$$

where $\phi(n)$ is the Euler totient function.

Simple and elegant proofs of (4) together with some generalizations have been published by various authors, since then; for example, by E. Cohen (cf. [5], Corollary 35; also cf. [7] and [8]), P. J. McCarthy (cf. [11], § 4), M. V. Subbarao [17], A. C. Vasu ,19] and P. Szüsz [18].

Another interesting special case of (2) which is a generalization of (4) that may be found in [8], [17] and [19] is the following (here $f(n) = n^k$, $h(n) = \mu(n)$):

$$(5) \qquad J_k(n) \sum_{\substack{d \mid n \\ (m,d)=1}} \frac{d^k}{J_k(d)} \mu\left(\frac{n}{d}\right) = \mu(n) \sum_{d \mid (m,n)} d^k \mu\left(\frac{n}{d}\right),$$

where $J_k(n)$ is the Jordan totient function (cf. [10], p. 147; also cf. [2] and [3]).

Two more interesting special cases of (2) which have been established by M. V. Subbarao (cf. [17], pp. 137-138) in 1965 are the following (here $f(n) = 2^{\omega(n)}$, $h(n) = \lambda(n)$, where $\omega(n)$ is the number of distinct prime factors of n and $\lambda(n)$ is the Liouville's function defined by $\lambda(n) = (-1)^{\omega(n)}$; and $f(n) = \phi(n)$, $h(n) = 1$ respectively):

$$(6) \qquad \sum_{\substack{d \mid n \\ (m,d)=1}} 2^{\omega(d)} \mu\left(\frac{n}{d}\right) = \mu(n) \sum_{d \mid (m,n)} 2^{\omega(d)} \lambda\left(\frac{n}{d}\right),$$

$$(7) \qquad n \sum_{\substack{d \mid n \\ (m,d)=1}} \frac{\phi(d)}{d} \mu\left(\frac{n}{d}\right) = \mu(n) \sum_{d \mid (m,n)} \phi(d) .$$

In this note we raise some questions relating to the functional equation (2). We also generalize (2) and raise similar questions relating to the generalized functional equation and discuss them in some particular cases. Finally, we prove a Theorem which characterizes the solutions of the generalized functional equation. Our first question is the following:

Problem 1. Given a completely multicative function f, characterize all multiplicative functions h for which the functional equation (2) is satisfied, where $F(n)$ is defined by (1).

Let $\mathscr{C}$ denote the set of all completely multiplicative arithmetical functions and $\mathscr{M}$ denote the set of all multiplicative arithmetical functions. It is clear that $\mathscr{C}$ is a subset of $\mathscr{M}$. In Problem 1 above, we asked to characterize all $h \in \mathscr{M}$ for which the functional equation (2) is satisfied whenever a $f \in \mathscr{C}$ is given. In the first place, one might consider the validity of the following converse statement of E. Cohen's theorem: Given a $f \in \mathscr{C}$ satisfying (2), then it would follow that h is independent of f and $h(n)$

$= \mu(n)\, g(n)$, where $g \in \mathscr{M}$. This is not true as can be seen from the example: Given a $f \in \mathscr{C}$, let $\mathscr{P}$ denote the set of all primes for which $f(p) \neq 0$. Define h to be an element of $\mathscr{M}$ such that $h(p^\alpha) = 0$ for $\alpha \geqslant 2$ and $h(p)$ can be arbitrary, whenever $p \in \mathscr{P}$; otherwise, that is, if $p \notin \mathscr{P}$, $h(p^\alpha)$ can be arbitrary for each positive integer α. It can be verified that this pair (f, h) satisfies the functional equation (2), although h is neither independent of f nor of the form μg, where $g \in \mathscr{M}$.

In the second place, it might appear from E. Cohen's theorem referred to above and Problem 1 that a necessary condition for (2) to be satisfied is that $f \in \mathscr{C}$. This is not true as can be seen from the identities (6) or (7), since in either case $f \in \mathscr{M} - \mathscr{C}$. So, our next question is the following:

Problem 2. Determine all pairs (f, h) of arithmetical functions for which the functional equation (2) is satisfied, where $f \in \mathscr{M}$ and $h \in \mathscr{M}$.

Remark 1. We prove (see Remark 3 and the Theorem below) that any pair (f, h) of arithmetical functions satisfies the functional equation (2) if and only if the corresponding F (defined by (1)) has the property

$$F(p^\alpha) f(p^{\alpha-1}) = f(p^\alpha) F(p^{\alpha-1}), \tag{8}$$

for all primes p and all integers $\alpha \geqslant 2$.

We note that if $f \in \mathscr{C}$ and $h = \mu g$, where $g \in \mathscr{M}$, then the corresponding F has the property (8). Also, the example given above, to show that the converse of E. Cohen's theorem is false, posesses the property (8).

We now generalize the functional equation (2) and raise similar questions as above relating to the generalized functional equation and discuss them in some particular cases: For each positive integer n, let $A(n)$ be a non-empty set of positive divisors of n. If f and h are arithmetical functions, let us define a new arithmetical function F_A by

$$F_A(n) = \sum_{d \in A(n)} f(d)\, h\left(\frac{n}{d}\right). \tag{9}$$

This furnishes a binary operation on the set $\mathscr{A}$ of arithmetical functions, which is called the arithmetical convolution A. Two examples of arithmetical convolutions are the classical Dirichlet convolution D, where $D(n)$ is the set of all positive divisors of n (see (1) above), and the unitary convolution U, where $U(n)$ is the set of all positive divisors of n such that $(d, n/d) = 1$. We write $d \,||\, n$ to mean that $d \in U(n)$. The unitary convolution has been discussed in detail by E. Cohen [9] in 1960.

An arithmetical convolution A is called regular if (i) $\mathscr{A}$ is a commutative ring with respect to addition and the convolution A, (ii) $f \in \mathscr{M}$ and $h \in \mathscr{M}$ implies F_A (defined by (9)) is an element of $\mathscr{M}$ and (iii) the function e defined by $e(n) = 1$ for all n has an inverse μ_A in the ring $\mathscr{A}$ and $\mu_A(n) = 0$ or -1 whenever n is a prime power. The convolutions D and U are regular. In 1963 W. Narkiewicz introduced the concept of regular arithmetical convolutions and characterized them in terms of the sets $A(n)$ (cf. [15], Theorems I and II). In particular, he showed that a regular arithmetical convolution is completely determined by the sets $A(p^\alpha)$ for all prime powers $p^\alpha > 1$, and that for each such prime power there is a positive divisor t of α such that

$$A(p^\alpha) = \{1, p^t, p^{2t}, \ldots, p^{st}\}, \quad st = \alpha,$$

and for $1 \leqslant j \leqslant \alpha/t$, $A(p^{jt}) = \{1, p^t, \ldots, p^{jt}\}$. The integer t is called the *type* of p^α (cf. [15], p. 87 or cf. [13], p. 3) and is denoted by $\tau_A(p^\alpha)$. Note that for all $p^\alpha > 1$, $\tau_D(p^\alpha) = 1$ and $\tau_U(p^\alpha) = \alpha$.

Let A be a regular arithmetical convolution. The function $\mu_A \in \mathscr{M}$ and for every prime p and positive integer α,

$$\mu_A(p^{(s-i)t}) = \begin{cases} 1 & \text{if} \quad i = s \\ -1 & \text{if} \quad i = s-1 \\ 0 & \text{if} \quad i \leqslant s-2\,. \end{cases} \tag{10}$$

Let $\phi_A(n)$ denote the number of integers x such that $1 \leqslant x \leqslant n$ and $(x, n)_A = 1$. It has been shown that $\phi_A \in \mathscr{M}$ and the properties of μ_A and ϕ_A have been discussed in detail by P. J. McCarthy [13]. We have $\mu_U = \mu^*$ and $\phi_U = \phi^*$, where μ^* and ϕ^* are the unitary analogues of μ and ϕ defined and discussed by E. Cohen [9].

Let $(m, n)^*$ denote the greatest divisor of m which is a unitary divisor of n. In 1962, P. J. McCarthy (cf. [12], eg. (18)) established that the functional equation

$$F_U(n) \sum_{\substack{d \| n \\ (m,d)=1}} \frac{f(d)}{F_U(d)} \mu^*\left(\frac{n}{d}\right) = \mu^*(n) \sum_{d \| (m,n)} f(d)\, h\left(\frac{n}{d}\right) \tag{11}$$

is satisfied when $f \in \mathscr{M}$ and $h(n) = \mu^*(n)\, g(n)$, where $g \in \mathscr{M}$, under the assumption that $f(p^\alpha) - g(p^\alpha) \neq 0$ for every prime p and every positive integer α.

An interesting special case of (11), namely the unitary analogue of the Brauer-Rademacher identity (4), has been stated by P. J. McCarthy (cf.

[12], p. 56), which has also been proved independently by E. L. Shader [16] is the following (here $f(n) = n$, $h(n) = \mu^*(n)$):

$$(12) \qquad \phi^*(n) \sum_{\substack{d\|n \\ (m,d)=1}} \frac{d}{\phi^*(d)} \mu^*\left(\frac{n}{d}\right) = \mu^*(n) \sum_{d\|(m,n)_*} d\mu^*\left(\frac{n}{d}\right)$$

Another interesting special case of (11), which has been stated by P. J. McCarthy (cf. [12], p. 56) is the following (here $f(n) = n$, $h(n) = 1$):

$$(13) \qquad \sigma^*(n) \sum_{\substack{d\|n \\ (m,d)=1}} \frac{d}{\sigma^*(d)} \mu^*\left(\frac{n}{d}\right) = \mu^*(n) \sum_{d\|(m,n)} d$$

where $\sigma^*(n)$ is the sum of all the unitary divisors of n.

We are now in a position to pose the following general questions:

Problem 3. Suppose A is any regular arithmetical convolution other than U. Given a $f \in \mathscr{C}$, characterize all $h \in \mathscr{M}$ for which the functional equation

$$(14) \qquad F_A(n) \sum_{d \in A((m,n)_A)} \frac{f(d)}{F_A(d)} \mu_A\left(\frac{n}{d}\right) = \mu_A(n) \sum_{d \in A((m,n)_A)} f(d)\, h\left(\frac{n}{d}\right)$$

is satisfied, where $(m, n)_A$ denotes the greatest divisor of m which belongs to $A(n)$.

Remark 2. In case $A = U$, any pair (f, h), where $f \in \mathscr{M}$ and $h \in \mathscr{M}$ satisfies the functional equation (14), since in this case (14) turns out to be (11) which has been shown to be satisfied by any pair $(f, \mu^* g)$, where $f \in \mathscr{M}$, $g \in \mathscr{M}$. It may be noted that since $\mu^*(n) \neq 0$ for all n and $\mu^* \in \mathscr{M}$, given any $h \in \mathscr{M}$, we can write $h = \mu^* g$, where $g \in \mathscr{M}$ which is defined by $g(n) = h(n)/\mu^*(n)$ for all n.

Problem 4. If A is any regular arithmetical convolution other than U, then determine all pairs (f, h) of arithmetical functions for which the functional equation (14) is satisfied, where $f \in \mathscr{M}$ and $h \in \mathscr{M}$.

Remark 3. If $f(n) = n$ and $h(n) = \mu_A(n)$, then $F_A(n)$ becomes $\phi_A(n)$ (cf. [13], Corollary 1.1) and it has been established by K. Nageswara Rao (cf. [14], Theorem 4.8) that this pair (f, h) satisfies the functional equation (14) by proving the following identity which is the A-analogue of the Brauer-Rademacher identity:

$$(15) \qquad \phi_A(n) \sum_{d \in A((m,n)_A)} \frac{n}{\phi_A(n)} \mu_A\left(\frac{n}{d}\right) = \mu_A(n) \sum_{d \in A((m,n)_A)} d\mu_A\left(\frac{n}{d}\right).$$

However, it has not been established by anybody so far that the functional equation (14) is satisfied by any pair $(f, \mu_A g)$, where $f \in \mathscr{C}$ and $g \in \mathscr{M}$, for every regular arithmetical convolution A. In a way, this is an expected result, since as we mentioned above that this result has been proved by E. Cohen and P. J. McCarthy in cases $A = D$ and $A = U$ respectively. We now prove a Theorem from which it follows as a particular case, the above expected result (proved in Corollary below) and the result stated in Remark 1 by taking $A = D$ in the Theorem.

THEOREM. *Suppose A is any regular arithmetical convolution other than U. Then any pair (f, h) of arithmetical functions satisfies the functional equation (14) if and only if the corresponding F_A (defined by (9)) has the property*

$$F_A(p^\alpha) f(p^{\alpha-t}) = f(p^\alpha) F_A(p^{\alpha-t}), \tag{16}$$

for all primes p and all integers $\alpha \geqslant 2$, where $t = \tau_A(p^\alpha)$.

Proof. Since A is a regular arithmetical convolution, the functions appearing on the two sides of (14) are multiplicative in both the arguments m and n. That is, if $L(m, n)$ and $R(m, n)$ denote the expressions on the left and right sides of (14), then $L(m_1 m_2, n_1 n_2) = L(m_1, n_1) L(m_2, n_2)$ and $R(m_1 m_2, n_1 n_2) = R(m_1, n_1) R(m_2, n_2)$, whenever $(m_1 n_1, m_2 n_2) = 1$ Hence it suffices to discuss (14) when m and n are powers of the same prime, say $m = p^\beta$ and $n = p^\alpha$, where $\alpha \geqslant 2$. Since $t = \tau_A(p^\alpha)$ and $st = \alpha$, we have $(m, n)_A = p^{jt}$, where j is the greatest integer $\leqslant s$ such that $p^{jt} \mid p^\beta$. Since $A \neq U$, we have $t \neq \alpha$, so that $1 \leqslant t \leqslant \alpha - 1$ and $s \geqslant 2$. Also, if $t \leqslant \beta$, then $j \geqslant 1$ and if $t > \beta$, then $j = 0$. We have by (10),

$$\begin{aligned}
L(p^\beta, p^\alpha) &= F_A(p^\alpha) \sum_{d \in A(p^{jt})} \frac{f(d)}{F_A(d)} \mu_A\left(\frac{p^\alpha}{d}\right) \\
&= F_A(p^\alpha) \sum_{i=0}^{j} \frac{f(p^{it})}{F_A(p^{it})} \mu_A(p^{(s-i)t}) \\
&= \begin{cases} F_A(p^\alpha)\, \mu_A(p^{st}), & \text{if } \ t > \beta \\ F_A(p^\alpha) \left[\dfrac{f(p^{jt})}{F_A(p^{jt})} \mu_A(p^{(s-j)t}) + \dfrac{f(p^{(j-1)t})}{F_A(p^{(j-1)t})} \mu_A(p^{(s-j+1)t}) \right], & \text{if } \ t \leqslant \beta \end{cases} \\
&= \begin{cases} 0, & \text{if } \ t > \beta \\ F_A(p^\alpha) \left[\dfrac{f(p^{st})}{F_A(p^{st})} - \dfrac{f(p^{(s-1)t})}{F_A(p^{(s-1)t})} \right], & \text{if } \ t \leqslant \beta \end{cases}
\end{aligned}$$

and

$$R(p^{\beta}, p^{\alpha}) = \mu_A(p^{st}) \sum_{d \in A(pj^t)} f(d)\, h\left(\frac{n}{d}\right) = 0\,, \text{ since } s \geqslant 2\,.$$

Now, it is easy to see that $L(p^{\beta}, p^{\alpha}) = R(p^{\beta}, p^{\alpha})$ if and only if (16) holds. Thus the Theorem is proved.

COROLLARY *If $f \in \mathscr{C}$ and $h = \mu_A g$, where $g \in \mathscr{M}$, then the pair (f, h) satisfies the functional equation (14).*

Proof. We have

$$F_A(p^{\alpha}) f(p^{\alpha-t}) = \left\{ \sum_{d \in A(p^{st})} f(d)\, \mu_A(p^{st}/d)\, g(p^{st}/d) \right\} f(p^{(s-1)t})$$

$$= \left\{ \sum_{i=0}^{s} f(p^{it})\, \mu_A(p^{(s-i)t})\, g(p^{(s-i)t}) \right\} f(p^{(s-1)t})$$

$$= \{ f(p^{st}) - f(p^{(s-1)t})\, g(p^t) \} f(p^{(s-1)t})$$

$$= f(p^{(2s-1)t}) - f(p^{(2s-2)t})\, g(p^t)\,, \tag{17}$$

since $f \in \mathscr{C}$. Also,

$$f(p^{\alpha}) F_A(p^{\alpha-t}) = f(p^{st}) \left\{ \sum_{i=0}^{s-1} f(p^{it})\, \mu_A(p^{(s-1-i)t})\, g(p^{(s-1-i)t}) \right\}$$

$$= f(p^{st}) \{ f(p^{(s-1)t}) - f(p^{(s-2)t})\, g(p^t) \}$$

$$= f(p^{(2s-1)t}) - f(p^{(2s-2)t})\, g(p^t)\,. \tag{18}$$

Now, from (17) and (18), we see that (16) holds for all primes p and all integers $\alpha \geqslant 2$, where $t = \tau_A(p^{\alpha})$. Hence the Corollary follows in virtue of the Theorem.

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