MAPS BETWEEN CLASSIFYING SPACES

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by J. F. ADAMS

In what follows, G and G' will be compact Lie groups; BG and BG' will be their classifying spaces; and I want to study the classification of maps

$$f : BG \to BG' .$$

What happens may be described in general terms; BG has a very rich and a very rigid structure, and the effect of this is that there are very few maps compared with what one might expect.

One can illustrate this by looking at a classical example. Take

$$G = G' = S^3 = Sp(1) = SU(2) = Spin(3) .$$

Then

$$BG = BG' = \mathbf{H}P^\infty ,$$

the infinite-dimensional projective space over the quaternions. Its cohomology ring is a polynomial algebra:

$$H^*(\mathbf{H}P^\infty; \mathbf{Z}) = \mathbf{Z}[x], \ x \in H^4 .$$

For any map

$$f: \mathbf{H}P^\infty \to \mathbf{H}P^\infty$$

we must have

$$f^*x = dx$$

for some $d \in \mathbf{Z}$. We call d the *degree* of f.

PROPOSITION 1. *The integers d which arise as the degrees of maps $f : \mathbf{H}P^\infty \to \mathbf{H}P^\infty$ are precisely 0 and the odd squares.*

To prove that the degree is necessarily a square k^2 is not hard; there is a choice of methods. To prove that k is either zero or odd one uses symplectic K-theory. By far the most substantial part of the proof is the construction of maps with the degrees stated; this is due to Sullivan [5].

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In particular, most of the maps $f : \mathbf{H}P^\infty \to \mathbf{H}P^\infty$ constructed by Sullivan are not of the form $B\theta$ for any homomorphism $\theta : S^3 \to S^3$ of Lie groups; for the integers d which arise as the degrees of maps $B\theta$ are precisely 0 and 1.

I have stated Proposition 1 in terms of ordinary cohomology. However, a theory which relies wholly on ordinary cohomology cannot be expected to work in a convenient and satisfactory way when the group G is not connected. At this point I should explain that my renewed interest in this subject was stimulated by conversations with C. B. Thomas. The direction of his work may be seen from [6]; and in his work the group G is finite. So I will try to cover the case in which G is not connected.

The appropriate measure is to classify maps $f : BG \to BG'$ according to the induced map of K-theory

$$f^* : K(BG) \leftarrow K(BG').$$

(When G is connected this gives the same classification as that in [1].) Here I recall that $K(X)$ means the generalised cohomology theory of Grothendieck-Atiyah-Hirzebruch; for our purposes we should use representable K-theory,

$$K(X) = [X, \mathbf{Z} \times BU],$$

where $[X, Y]$ means the set of homotopy classes of maps from X to Y. This is the best definition when X is an infinite complex, and BG is usually infinite.

The use of K-theory would hardly be profitable if we had no means of computing $K(BG)$; fortunately we do. Let RG be the representation ring of the compact Lie group G. If $\theta : G \to U(n)$ is a representation, we can form the composite

$$BG \xrightarrow{B\theta} BU(n) \subset n \times BU \subset \mathbf{Z} \times BU,$$

and this composite gives an element $\alpha(\theta) \in K(BG)$; this construction defines a homomorphism of rings

$$\alpha : RG \to K(BG).$$

PROPOSITION 2 [2, 3, 4]. *The map* α *induces an isomorphism*

$$\hat{\alpha} : \hat{RG} \to K(BG).$$

Here $\hat{RG}$ means the completion of RG with respect to a topology which one has to describe. Consider the map of groups $1 \to G$. This induces the "augmentation" map

$$\varepsilon : RG \to R1 = \mathbf{Z},$$

which assigns to each representation $\theta : G \to U(n)$ its dimension n. The augmentation ideal $I \subset RG$ is defined to be Ker ε. The topology in question is that in which the neighbourhoods of 0 are the powers I^n of the augmentation ideal.

This means that when G is a given finite group, general results expressed in terms of $K(BG)$ can be interpreted by calculations with the character table of G; and these are calculations which algebraists prefer to homological calculations.

For example, take $G = SL(2, 5)$, the binary icosahedral group, and take $G' = SU(2)$. We want to know the possible values for

$$f^* : K\big(BSL(2,5)\big) \leftarrow K\big(BSU(2)\big).$$

Now the composite

$$BSU(2) \subset 2 \times BU \subset \mathbf{Z} \times BU$$

defines an element $i_2 \in K\big(BSU(2)\big)$; and it is sufficient to know $f^* i_2$, because this determines $f^* x$ for every other element $x \in K\big(BSU(2)\big)$. So we wish to know the composite

$$BSL(2,5) \xrightarrow{f} BSU(2) \xrightarrow{i_2} \mathbf{Z} \times BU.$$

In order to describe it, let

$$i : SL(2,5) \to SU(2)$$

be a fixed choice of one of the two standard embeddings. Then the general results I shall present specialise as follows.

PROPOSITION 3. (a) *For any map*

$$f : BSL(2,5) \to BSU(2)$$

the composite

$$BSL(2,5) \xrightarrow{f} BSU(2) \xrightarrow{i_2} \mathbf{Z} \times BU$$

is equal to

$$BSL(2,5) \xrightarrow{Bi} BSU(2) \xrightarrow{\psi^k} \mathbf{Z} \times BU$$

for some integer k.

(b) *Moreover, two composites*

$$BSL(2,5) \xrightarrow{Bi} BSU(2) \xrightarrow{\psi^k} \mathbf{Z} \times BU$$
$$BSL(2,5) \xrightarrow{Bi} BSU(2) \xrightarrow{\psi^l} \mathbf{Z} \times BU$$

are equal if and only if they have the same second Chern class, that is if and only if $k^2 \equiv l^2 \bmod 120$.

Roughly speaking, this result says that to the eyes of K-theory, any map $f : BSL(2, 5) \to BSU(2)$ looks like one of the examples constructed by Milnor. Here I recall that the examples constructed by Milnor are the composites

$$BSL(2,5) \xrightarrow{Bi} BSU(2) \xrightarrow{f'} BSU(2),$$

where f' is a map of degree k^2 (see Proposition 1). Of course one can only construct such an example when k is odd (or zero). It is likely that a "best possible" version of Proposition 3 would specify that k^2 has to be odd or zero mod 120; however, for C. B. Thomas' purposes such a result would be no more useful than the one given.

The examples of Milnor show in particular that even when G is finite, there exist maps $f : BG \to BG'$ which are not of the form $B\theta$ for any homomorphism $\theta : G \to G'$. In fact, with the notation of Proposition 3, the maps of the form $B\theta$ have invariants $k^2 \equiv 0, 1, 49 \bmod 120$. (There is another embedding of $SL(2, 5)$ in $SU(2)$ besides the one which was chosen as i; this gives a map $B\theta$ with invariant $k^2 \equiv 49 \bmod 120$.)

I mention that the most important properties of $SL(2, 5)$ which are used in proving Proposition 3 also hold for the other finite groups which can act freely on spheres. This gives grounds for hoping that the method applies well to such groups.

One may note that Proposition 3 gives a classification into a finite list of possibilities (corresponding to the residue classes k^2 mod 120). This behaviour is general; when G is finite the theorems to follow always lead to a finite list of possibilities.

We now address the problem of formulating some general theorems. Suppose given a map $f : BG \to BG'$; then we can form the following diagram.

$$\begin{tikzcd} K(BG) & K(BG') \arrow[l, "f^*"'] \\ RG \arrow[u, "\alpha"] & RG' \arrow[u, "\alpha'"'] \end{tikzcd}$$

It would be very gratifying if we could prove that $f^*\mathrm{Im}\,\alpha' \subset \mathrm{Im}\,\alpha$; this would place a very substantial restriction on f, and would tend to reduce the classification to pure algebra. Unfortunately it is not true in general.

Example 4. There is a compact Lie group G and a map

$$f\colon BG \to BU(2)$$

such that the composite

$$BG \xrightarrow{f} BU(2) \subset 2 \times BU \subset \mathbf{Z} \times BU$$

is an element $x \in K(BG)$ with $x \notin \operatorname{Im} \alpha$.

However, the example does have the property that $2x \in \operatorname{Im} \alpha$.

It is now more or less clear that we have to replace $\operatorname{Im} \alpha$ by something a bit more subtle. In general terms, we may regard the elements $x \in \operatorname{Im} \alpha \subset K(BG)$ as ones which can be constructed by finitistic, algebraic means; we may regard general elements $x \in K(BG)$ as constructed by infinitistic, topological means such as completion (see Proposition 2). There are examples, such as Example 4, of elements which can be constructed by finitistic, algebraic means although they are not in $\operatorname{Im} \alpha$. Therefore I propose to define a subset $\bar{R}G$ which we think of as "all the elements x which can be constructed by finitistic algebraic means", so that $\operatorname{Im} \alpha \subset \bar{R}G \subset K(BG)$.

At this stage I should apologise to the reader; in preparing this text I have not had time to write down all the proofs which I would like to write down. I will continue to give the statements as I made them in my lecture, because I think they are more likely to be true than false; but the reader may well treat them with caution till he sees proofs in print.

The definition which I gave in my lecture read as follows: an element $x \in K(BG)$ lies in $\bar{R}G$ if and only if there exists an integer $n \neq 0$ such that $nx \in \operatorname{Im} \alpha$. This has the effect of throwing the torsion subgroup of $K(BG)$ into $\bar{R}G$, but I trust that this torsion subgroup is zero. So the "finitistic, algebraic means" which are allowed, in addition to those used in constructing $\operatorname{Im} \alpha$, include division by non-zero integers. I hope that this definition is good; but if I should have trouble with my proofs, I shall fall back on an earlier definition of $\bar{R}G$ which is longer and more complicated to explain.

It is now fairly clear what result I seek.

THEOREM 5. *Let G and G' be compact Lie groups, and let $f\colon BG \to BG'$ be a map; then*

$$f^*\colon K(BG) \leftarrow K(BG')$$

carries $\bar{R}G'$ into $\bar{R}G$.

The introduction of $\bar{R}G$ means that we need subsidiary results to remove it again in favourable cases.

PROPOSITION 6. *If G is finite then* $\bar{R}G = \operatorname{Im} \alpha$.

PROPOSITION 7. *If G is a compact Lie group and its group of components, $\pi_0 G$, is the union of its Sylow subgroups, then $\alpha : RG \to K(BG)$ is mono and* $RG = \operatorname{Im} \alpha$.

Of course, neither Proposition 6 nor Proposition 7 applies to the group G used in Example 4; for that one, G is not finite and $\pi_0 G$ is not the union of its Sylow subgroups.

The only reasonable way to prove a result like Theorem 5 is to characterise $\bar{R}G$ in some topological way which is preserved by induced maps f^*. For this purpose I need the exterior power operations. It is also convenient to introduce the total exterior power λ_t; this is given by

$$\lambda_t(x) = \sum_{i=0}^{\infty} \lambda^i(x)\, t^i ;$$

it lies in the ring of formal power series $K(BG)\,[[t]]$, where t is a new variable introduced for the purpose.

THEOREM 8. *Suppose G is a compact Lie group and $x \in K(BG)$ is an element such that $\lambda_t(x)$ is a polynomial in t, i.e. $\lambda^i(x) = 0$ for i sufficiently large. Then $x \in \bar{R}G$.*

Proof of Theorem 5 from Theorem 8. Suppose $x \in \bar{R}G'$. Then there exists $n \neq 0$ such that $nx \in \operatorname{Im} \alpha'$; say $nx = \alpha'(y-z)$ for some

$$y : G' \to U(q), \quad z : G' \to U(r).$$

Then

$$\lambda^i y = 0 \text{ for } i > q, \quad \lambda^i z = 0 \text{ for } i > r.$$

Therefore

$$\lambda^i(f^*\alpha' y) = 0 \text{ for } i > q, \quad \lambda^i(f^*\alpha' z) = 0 \text{ for } i > r.$$

By Theorem 8,

$$f^*\alpha' y \in \bar{R}G, \quad f^*\alpha' z \in \bar{R}G.$$

So

$$f^*\alpha'(y-z) \in \bar{R}G,$$

that is

$$nf^*x \in \bar{R}G.$$

Hence

$$f^*x \in \bar{R}G.$$

This completes the proof.

If G is finite we can make Theorem 8 more precise.

PROPOSITION 9. *Assume* G *finite. In order that* $x \in K(BG)$ *should lie in* $\operatorname{Im} \alpha$, *it is necessary and sufficient that* $\lambda_t(x)$ *should be a rational function of* t.

Here a formal power-series $f(t)$ is called a "rational function of t" if it can be written as the quotient $g(t)/h(t)$ of two polynomials $g(t)$ and $h(t)$, with $h(t)$ invertible in $K(BG)[[t]]$.

In Proposition 9, the "necessity" is obvious and requires no assumptions on G. The "sufficiency" does require assumptions.

Example 10. There is a compact Lie group G and an element $x \in K(BG)$ such that $\lambda_t(x)$ is a rational function of t but $x \notin \bar{R}G$.

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