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If  $G$  is the group  $SO_2$  of rotations of  $S^1$ , then  $H(L_{S^1}; SO_2)$  is a model for  $C^*(L_{S^1}; SO_2)$ . It is generated by  $u$  and by an element  $e$  of degree 2 represented by

$$e(f, g) = \int_0^1 \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} dx$$

The only relation is  $eu = 0$ .

## 2. CONNECTION WITH FOLIATIONS

Let me indicate very briefly the relation with characteristic classes of flat bundles (cf. [12]).

$H^*(L_M, G)$  could also be interpreted as the differentiable cohomology of a suitable differentiable category (for more informations see [4] and [15]).

We consider on the product  $X \times M$  of a smooth manifold  $X$  with  $M$  a smooth foliation  $F$  whose leaves have the same dimension as  $X$  and cut each fibers  $\{x\} \times M$  transversally.

To such a foliation is naturally associated a continuous  $DG$ -algebra map

$$\chi_F : C^*(L_M) \rightarrow \Omega_X$$

where  $\Omega_X$  is the  $DG$ -algebra of differential forms on  $X$ . In fact there is a bijection between such morphisms and foliations  $F$  as above.

Passing to cohomology, we get the characteristic map

$$H^*(L_M) \rightarrow H^*(X; R)$$

If we replace the trivial bundle by a bundle  $E$  with fiber  $M$ , base space  $X$  and structural group  $G$ , then for a foliation  $F$  on  $E$  complementary to the fibers, we still get a morphism

$$\chi_F : C^*(L_M; G) \rightarrow \Omega_X$$

hence a characteristic homomorphism

$$H^*(L_M, G) \rightarrow H^*(X; R)$$

Denoting by  $BG$  the classifying space for  $G$ -bundles, we also have the usual characteristic map  $H^*(BG; R) \rightarrow H^*(X; R)$ . This map factorizes

through a map  $H^*(BG; R) \rightarrow H^*(L_M; G)$  so that we get a commutative diagram

$$\begin{array}{ccc}
 & & H^*(L_M; G) \\
 & \nearrow & \downarrow \\
 H^*(BG; R) & & \\
 & \searrow & \\
 & & H^*(X; R)
 \end{array}$$

So it is important to compute the map  $H^*(BG; R) \rightarrow H^*(L_M; G)$ . When  $G$  is a compact connected Lie group, then  $H^*(BG; R)$  is the algebra  $I(G)$  of invariant polynomials on the Lie algebra of  $G$ , and the map from  $I(G)$  to  $C^*(L_M; G)$  is given by a  $G$ -connexion in  $C^*(L_M)$  (cf. [5]).

In the example above, namely  $M = S^1$  and  $G = SO_2$ , then  $H^*(BSO_2)$  is a polynomial algebra in a generator of degree 2, the Euler class, which is mapped on a non zero multiple of  $e$ .

### 3. THE FORMAL VECTOR FIELDS AND THE DIAGONAL COMPLEX

Given a point  $x$  on  $M$ , we can consider the Lie algebra  $L_M^x$  of infinite jets at  $x$  of vector fields on  $M$  with the quotient topology. It is isomorphic to the Lie algebra  $\mathfrak{a}_n$  of formal vector fields  $\sum v_i(x) \partial/\partial x^i$  in  $R^n$ , where the  $v_i(x)$  are formal power series in the coordinates  $x^1, \dots, x^n$ .

The natural map  $L_M \rightarrow L_M^x$  associating to a vector field its jet at  $x$  gives a  $DG$ -algebra morphism

$$C^*(L_M^x) \rightarrow C^*(L_M)$$

where  $C^*(L_M^x)$  is the algebra of multilinear alternate forms on  $L_M^x$  depending only on finite order jets.

The first and most important step in the work of Gelfand-Fuks was the complete determination of the cohomology  $H^*(\mathfrak{a}_n)$  of the topological Lie algebra of formal vector fields on  $R^n$ .

**THEOREM 1.** (Gelfand-Fuks [8], [9]). *Let  $E(h_1, \dots, h_n)$  be the exterior algebra on generators  $h_i$  of degree  $2i-1$  and let  $R[c_1, \dots, c_n]_{2n}^\wedge$  be the quotient of the polynomial algebra in generators  $c_i$  of degree  $2i$  by the ideal of elements of degree  $> 2n$ .*