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Kapitel: 2. Connection with foliations

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If G is the group SO_2 of rotations of S^1, then $H(L_{S^1}; SO_2)$ is a model for $C^*(L_{S^1}; SO_2)$. It is generated by u and by an element e of degree 2 represented by

$$e(f, g) = \int_0^1 \begin{vmatrix} f, & g \\ f', & g' \end{vmatrix} dx$$

The only relation is $e\,u = 0$.

2. Connection with foliations

Let me indicate very briefly the relation with characteristic classes of flat bundles (cf. [12]).

$H^*(L_M, G)$ could also be interpreted as the differentiable cohomology of a suitable differentiable category (for more informations see [4] and [15]).

We consider on the product $X \times M$ of a smooth manifold X with M a smooth foliation F whose leaves have the same dimension as X and cut each fibers $\{x\} \times M$ transversally.

To such a foliation is naturally associated a continuous DG-algebra map

$$\chi_F : C^*(L_M) \to \Omega_X$$

where Ω_X is the DG-algebra of differential forms on X. In fact there is a bijection between such morphisms and foliations F as above.

Passing to cohomology, we get the characteristic map

$$H^*(L_M) \to H^*(X; R)$$

If we replace the trivial bundle by a bundle E with fiber M, base space X and structural group G, then for a foliation F on E complementary to the fibers, we still get a morphism

$$\chi_F : C^*(L_M; G) \to \Omega_X$$

hence a characteristic homomorphism

$$H^*(L_M, G) \to H^*(X; R)$$

Denoting by BG the classifying space for G-bundles, we also have the usual characteristic map $H^*(BG; R) \to H^*(X; R)$. This map factorizes through a map $H^*(BG; R) \to H^*(L_M; G)$ so that we get a commutative diagram

$$\begin{array}{ccc} & \nearrow & H^*(L_M; G) \\ H^*(BG; R) & & \downarrow \\ & \searrow & H^*(X; R) \end{array}$$

So it is important to compute the map $H^*(BG; R) \to H^*(L_M; G)$. When G is a compact connected Lie group, then $H^*(BG; R)$ is the algebra $I(G)$ of invariant polynomials on the Lie algebra of G, and the map from $I(G)$ to $C^*(L_M; G)$ is given by a G-connexion in $C^*(L_M)$ (cf. [5]).

In the example above, namely $M = S^1$ and $G = SO_2$, then $H^*(BSO_2)$ is a polynomial algebra in a generator of degree 2, the Euler class, which is mapped on a non zero multiple of e.

3. The formal vector fields and the diagonal complex

Given a point x on M, we can consider the Lie algebra L_M^x of infinite jets at x of vector fields on M with the quotient topology. It is isomorphic to the Lie algebra $\mathfrak{a}_n$ of formal vector fields $\sum v_i(x)\, \partial/\partial x^i$ in R^n, where the $v_i(x)$ are formal power series in the coordinates $x^1, \ldots, x^n$.

The natural map $L_M \to L_M^x$ associating to a vector field its jet at x gives a DG-algebra morphism

$$C^*(L_M^x) \to C^*(L_M)$$

where $C^*(L_M^x)$ is the algebra of multilinear alternate forms on L_M^x depending only on finite order jets.

The first and most important step in the work of Gelfand-Fuks was the complete determination of the cohomology $H^*(\mathfrak{a}_n)$ of the topological Lie algebra of formal vector fields on R^n.

Theorem 1. (Gelfand-Fuks [8], [9]). *Let* $E(h_1, \ldots, h_n)$ *be the exterior algebra on generators* h_i *of degree* $2i-1$ *and let* $R[c_1, \ldots, c_n]^{\wedge}_{2n}$ *be the quotient of the polynomial algebra in generators* c_i *of degree* $2i$ *by the ideal of elements of degree* $> 2n$.