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THEOREM 3. (Guillemin [10], Losik [17]). *$C^*_\triangle(L_M, \Omega_M)$ is a model for a bundle $E$ with fiber $F_n$, base space $M$, associated to the tangent bundle of $M$.*

*More precisely, a model for $C^*_\triangle(L_M, \Omega_M)$ is the DG-algebra $\Omega_M \otimes WU_n$ over $\Omega_M$, where*

$$d(1 \otimes c_i) = 0 \quad d(1 \otimes h_i) = 1 \otimes c_i - p_{i/2} \otimes 1$$

*where $p_{i/2}$ is zero if $i$ is odd and is a form representing the Pontrjagin class of $M$ of degree $2i$ if $i$ is even.*

Note that if a foliation $F$ on $X \times M$ transverse to the fibers $\{x\} \times M$ is given, one has a characteristic homomorphism

$$C^*(L_M, \Omega_M) \to \Omega_{X \times M}$$

One has also a morphism

$$WO_n \to C^*_\triangle(L_M, \Omega_M)$$

(or $WU_n \to C^*(L_M, \Omega_M)$ in case $M$ has trivial Pontrjagin classes) whose composition with the previous one is the usual characteristic homomorphism for the foliation $F$ (cf. [3], [12]).

## 4. MAIN THEOREM

THEOREM 1. *$C^*(L_M)$ is a model for the space $\Gamma$ of continuous sections of the bundle $E$ described in the theorem above.*

This result, first conjectured by Bott (and also Fuks), has been proved by several people (Bott-Segal[1]), Fuks-Segal, Haefliger [13], Ph. Trauber, and others).

Suppose that $G$ is a compact connected Lie group acting on $M$. Then it also acts on the bundle $E$ and on its space of sections. Let us denote by $\Gamma_G$ the total space of the bundle with fiber $\Gamma$ associated to the universal $G$-bundle with base space $BG$.

THEOREM 1′. *$C^*(L_M; G)$ is a model for the space $\Gamma_G$.*

The way I proved theorem 1 was to construct first a tentative algebraic model $A$ for $\Gamma$ following ideas of R. Thom [20] and D. Sullivan [18], and

[1]) Added on proof: *Topology 16* (1977), pp. 285-298.

a morphism of $A$ in $C^* (L_M)$. Then one proves directly that it induces an isomorphism in cohomology. The fact that $A$ is also a model for $\Gamma$ was proved in a similar way (cf. [14]).

When $M$ has a finite dimensional model, one can construct a model for $\Gamma$ which is finite dimensional in each degree, and with it one can make explicit calculations.

Note that the inclusion $C^*_\triangle (L_M, \Omega_M) \to C^* (L_M, \Omega_M)$ is a model for the evaluation map $\Gamma \times M \to E$ associating to a section $s$ and a point $x$ of $M$ the element $s(x)$ of $E$.

For computations along the lines of the spectral sequence of Gelfand-Fuks, see Cohen and Taylor [22].

The proof of theorem 1′ is very similar to the proof of theorem 1. In the next paragraph, we explain the construction of an algebraic model for $\Gamma_G$ suitable for computations. In § 6, we indicate briefly why this is a model for $\Gamma_G$.

## 5. Construction of an algebraic model for the space of sections of a fiber bundle ([20], [18], [13]).

As a guide, consider first the geometric situation. Let $p: E \to M$ be a fiber bundle with base space $M$, fiber $F$ and let $\Gamma$ be the space of continuous sections of $E$.

We have the commutative diagramm

1)
$$\begin{tikzcd}
M \times \Gamma \arrow[rr, "e"] \arrow[dr] \arrow[dd] & & E \arrow[dl, "p"] \\
& M \arrow[dd] & \\
\Gamma \arrow[dr] & & \\
& * &
\end{tikzcd}$$

where $e$ is the evaluation map associating to the point $x$ of $M$ and the section $s$ the point $s(x)$ of $E$. The other maps are natural projections ($*$ is a point).