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Autor: Haefliger, André
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which are models for the maps $\Gamma_M \to \Gamma'_M$ and $\Gamma'_{M,N} \to \Gamma'_M$. The first one is obvious and the second one is completely characterized by the map $WU_n \to WU_{n,p}$.

Now we get the differential on $\Lambda (x^j_\alpha, y^j_\lambda)$ by considering this algebra as the tensor product over $\Lambda (z^j_\alpha)$ of $\Lambda (x^j_\alpha, z^j_\lambda)$ with $\Lambda (y^j_\lambda)$.

One can make a similar construction using for A and B the DG-algebras Ω_M and Ω_N of differential forms on M and N. Of course one has to work again in more intrisic terms and use the C^∞-topology on Ω_M and Ω_N (compare with [13]). In this way one gets a DG-algebra which is also a model for $\Gamma_{M,N}$ (in fact one proves directly that it is a model for the DG-algebra constructed above), with a map in $C^* (L_{M,N})$ inducing an isomorphism in cohomology.

Summing up, we get the following result.

THEOREM. *Assume that the inclusion of N in M has a model which is a surjection of finite dimensional DG-algebras. One can construct explicitely a model for $C^* (L_{M,N})$ which is finite dimensional in each degree.*

Example. Suppose that M is the disk D^2 and N its boundary $\partial D^2 = S^1$. As the inclusion of $F_{2,1}$ in F_2 is homotopically trivial (equivalently the morphism $WU_2 \to WU_{2,1}$ is homotopic to zero), the bundle $\Gamma_{M,N} \to \Gamma'_{M,N}$ is trivial. WU_2 is a model for $S^5 \vee S^5 \vee S^7 \vee S^8 \vee S^8$ and $WU_{2,1}$ for $S^3 \vee S^3 \vee S^3 \vee S^4 \vee S^4$.

Hence $C^* (L_{D^2, \partial D^2})$ is a model for the space which is the product of the space of maps of S^1 in $S^3 \vee S^3 \vee S^3 \vee S^4 \vee S^4$ with the second loop space of $S^5 \vee S^5 \vee S^7 \vee S^8 \vee S^8$.

One can write down quite explicitely the minimal model for that space, but it is harder to compute the cohomology of the first factor. It has an infinite number of multiplicative generators.

10. Some other problems

1. As coefficient for the Gelfand-Fuks cochains, one might consider, instead of the field R with the trivial action of L_M, a topological L_M-algebra A. The problem is to find a model for the DG-algebra $C^* (L_M, A)$ of continuous multilinear alternate forms on L_M with values in A. The differential is defined by the usual formula involving the action of L_M on A.

For that case, results similar to the one mentionned in this report have been obtained by Fuks-Segal (unpublished) and by T. Tsujishita [21].

For instance, when A is the algebra of smooth functions on M on which L_M acts by Lie derivative, their result is as follows. As it is described in § 3, the bundle E over M has a fiber F_n which is itself a principal U_n-bundle. Let us fix a fiber $F_n^{\circ} \approx U_n$ of this bundle; as it is invariant by the structural group $O_n \subset U_n$ of E, we get a subbundle E_o of E with typical fiber F_n^o. Then $C^* (L_M, A)$ will be a model for the inverse image of E_o by the evaluation map $M \times \Gamma \to E$.

2. One of the most interesting problems is to know when, for a given class α in $H^* (L_M)$, there is a space X and a foliation F on $X \times M$ transverse to the fibers such that the image of α in $H^* (X)$ by the characteristic homomorphism (cf. 2) is non zero.

Very recent and remarkable results of Fuchs [23] show that this is the case for all classes coming from WSO_n. (For earlier partial results, see [4].) One might expect that his method will apply in general and show that the answer is affirmative for all classes in $H^* (L_M)$ (and also for the similar problem with $H^* (L_M; G)$).

There is also the problem of the possible continuous variations of characteristic classes for flat bundles which would be interesting to study (cf. [23]).

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