§2. Pseudo-convex Domains

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Since ψ_n and φ_n are both injective outside $\Delta_n \times \mathbf{C}$, we can replace Ω by $\Omega \cap (U_n, \psi_n)$ over $(D_n - \{|z| < r_n) \times \mathbf{C}$, (the intersection being relative to ψ_n) and by (U_n, ψ_n) over $\{|z| < r_n\} \times \mathbf{C}$. [See the shaded figure in the diagram above.]

This gives us a manifold X, and a map $\pi : X \to \mathbf{C}^2$, such that $\pi^{-1}(a)$ contains at most two points for any $a \in \mathbf{C}^2$. This is locally Stein; in fact the boundary of $\pi^{-1}(D_n \times \mathbf{C})$ is locally described by an inequality $\max(u, v) < 0$ where u, v are strongly plurisubharmonic. This is known to be sufficient to guarantee that $\pi^{-1}(D_n \times \mathbf{C})$ is Stein. Over a small neighbourhood U of $(0, 0)$, $\pi^{-1}(U)$ is isomorphic to the disjoint union $\bigcup_{n \geq 2} \psi_n^{-1}(U)$. Thus $\pi : X \to \mathbf{C}^2$ is locally Stein. However, X is not Stein. In fact, if

$$K = \{(z, w) \in \mathbf{C}^2 \mid z \text{ real}, \quad 0 \leqslant z \leqslant \tfrac{1}{2}, \quad |w| = 1\},$$

the envelope $L = (\pi^{-1}(K))^\wedge$ of the compact set $\pi^{-1}(K)$ has the property that $\pi(L)$ contains the discs

$$z = \frac{1}{n}, \ |w| \leqslant 1$$

for $n \geqslant 2$, so that L cannot be compact.

§ 2. Pseudo-convex Domains

Strongly pseudo-convex domains with smooth boundary in $\mathbf{C}^n$ have some very useful properties not shared by arbitrary bounded domains of holomorphy. Here are two such properties.

I. Let Ω be a strongly pseudo-convex domain with smooth boundary. Then Ω has a fundamental system of pseudo-convex neighbourhoods.

II. *Subelliptic estimates.*

We begin with a definition.

If $f \in L^2(\mathbf{R}^n)$, let $\hat{f}$ denote its Fourier transform, and, for a real $s \geqslant 0$, define $\|f\|_s$ by

$$\|f\|_s^2 = \int_{\mathbf{R}^n} |\hat{f}(\xi)|^2 (1 + |\xi|^2)^s \, d\xi$$

We set

$$\mathscr{H}^s(\mathbf{R}^n) = \{f \in L^2(\mathbf{R}^n) \mid \|f\|_s < \infty\}.$$

Let

$$\mathbf{R}^n_+ = \{(x_1, \dots, x_n) \in \mathbf{R}^n \mid x_n > 0\},$$

and define $\mathscr{H}^s(\overline{\mathbf{R}}^n_+)$ as the set of $g \in L^2(\mathbf{R}^n)$ for which there exists $f \in \mathscr{H}^s(\mathbf{R}^n)$ with $f = g$ on $\mathbf{R}^n_+$. We introduce a norm on $\mathscr{H}^s(\overline{\mathbf{R}}^n_+)$ by setting

$$\|g\|_s = \inf \|f\|_s,$$

the infimum being taken over those $f \in \mathscr{H}^s(\mathbf{R}^n)$ for which $f = g$ on $\mathbf{R}^n_+$.

This definition extends to compact manifolds with boundary (using local coordinates and a finite partition of unity).

If I, J are strictly increasing sequences of integers between 1 and n, $I = (i_1, \dots, i_p)$, $J = (j_1, \dots, j_q)$, we set

$$dz_I = dz_{i_1} \wedge \dots \wedge dz_{i_p}, \quad d\bar{z}_J = d\bar{z}_{j_1} \wedge \dots \wedge d\bar{z}_{j_q}.$$

Let Ω be a bounded open set in $\mathbf{C}^n$ with smooth boundary, and let ω be a C^∞ differential form on $\overline{\Omega}$. We can write ω uniquely in the form

$$\omega = \sum_{I, J} a_{IJ}\, dz_I \wedge d\bar{z}_J,$$

where I, J are strictly increasing sequences of integers between 1 and n and a_{IJ} are C^∞-functions on $\overline{\Omega}$. We define the s-norm of ω by

$$\|\omega\|_s^2 = \sum_{I, J} \|a_{IJ}\|_s^2.$$

If ω, ω' are differential forms of the same type (p, q),

$$\omega = \sum_{I, J} a_{IJ}\, dz_I \wedge d\bar{z}_J, \quad \omega' = \sum_{I, J} a'_{IJ}\, dz_I \wedge d\bar{z}_J,$$

we define a scalar product $< \omega, \omega' >$ by

$$< \omega, \omega' > = \sum_{I, J} \int_\Omega a_{IJ}\, \overline{a'_{IJ}}\, dv$$

(dv = Lebesgue measure)

and use this scalar product to define the adjoint $\bar{\partial}^*$ of the operator $\bar{\partial}$. Note that the condition that a C^∞-form on $\overline{\Omega}$ be in the domain of $\bar{\partial}^*$ is given by boundary conditions on the form.

Let Ω be a pseudoconvex domain in $\mathbf{C}^n$ with smooth boundary, and let $a \in \partial\Omega$. We say that the $\bar{\partial}$-Newmann problem is subelliptic at a for forms of type (p, q) if there exists a neighbourhood U of a in $\mathbf{C}^n$ and constants $C > 0$, $\varepsilon > 0$ such that

$$(**) \quad \|\omega\|_\varepsilon^2 \leqslant C\{\|\bar{\partial}\omega\|_0^2 + \|\bar{\partial}^*\omega\|_0^2 + \|\omega\|_0^2\}$$

for all forms of type (p, q) with compact support in $U \cap \bar{\Omega}$ which lie in the domain of $\bar{\partial}^*$.

One of the central results in the study of the behaviour of solutions of the equation $\bar{\partial} \varphi = \omega$ near the boundary in the following theorem (see Kohn [12]; also Princeton Mathematical Notes No. 19 by P. Greiner and E. Stein: *Estimates for the $\bar{\partial}$-Neumann Problem*, as well as the references given there).

THEOREM. *Let Ω be strongly pseudo-convex with smooth boundary. Then, for any $a \in \bar{\partial} \Omega$, any $q > 0$, $p \geqslant 0$, the $\bar{\partial}$-Neumann problem is subelliptic at a for forms of type (p, q).*

In fact, we may take $\varepsilon = \frac{1}{2}$ in (**) in this case, and this is best possible.

While this theorem, and the theorem about pseudo-convex neighbourhoods of $\bar{\Omega}$, fail to be true for pseudo-convex domains in general, it follows from results of Kohn [13] and Diederich-Fornaess [4] that they remain true if the boundary is real-analytic (without any hypotheses concerning the points where Ω is strongly pseudo-convex). We shall now describe the results of Kohn [13] and Diederich-Fornaess [4] in a little greater detail.

Let ρ be a real-valued C^∞-function on an open set U in $\mathbf{C}^n$, let $a \in U$, $\rho(a) = 0$. Suppose that $d\rho \neq 0$ on U. Then, the set

$$M = \{ x \in U \mid \rho(x) = 0 \}$$

is a real submanifold of U of dimension $2n - 1$. The complex hyperplane

$$T_a^{1,0}(M) = \{ (\zeta_1, \dots, \zeta_n) \in \mathbf{C}^n \mid \sum_{\nu=1}^{n} \frac{\partial \rho}{\partial z_\nu}(a) \cdot \zeta_\nu = 0 \}$$

is called the complex tangent space of M at a (it is the largest complex subspace of $\mathbf{C}^n$ contained in the real tangent space of M at a).

The Levi form of ρ at a is the restriction to $T_a^{1,0}(M)$ of the hermitian form

$$\sum_{\mu, \nu = 1}^{n} \frac{\partial^2 \rho}{\partial z_\mu \, \partial \bar{z}_\nu}(a) \cdot \zeta_\mu \cdot \bar{\zeta}_\nu .$$

We also call this the Levi form of M at a. Note that a change in the defining equation ρ of M merely multiplies this form by a non-zero real constant.

The null space of this Levi form is the set of vectors

$$N_a = \{ (\zeta_1, \dots, \zeta_n) \in T_a^{1,0}(M) \mid \sum_{\mu=1}^{n} \frac{\partial^2 \rho}{\partial z_\mu \, \partial \bar{z}_\nu}(a) \cdot \zeta_\mu = 0, \nu = 1, \dots, n \} .$$

Let X be a real analytic set in U with $a \in X$. We define the (Zariski) tangent space $T_a^{1,0}(X)$ of X at a as follows: let I_x be the ideal of germs at a of real-analytic functions that vanish on X in some neighbourhood of a. Then

$$T_a^{1,0}(X) = \left\{ L = \sum_{\nu=1}^{n} \zeta_\nu \frac{\partial}{\partial z_\nu} \in T_a^{1,0}(\mathbf{C}^n) \mid L(f) = 0, \forall f \in I_x \right\}.$$

If $X \subset M$, the integer

$$\inf_{a \in X} \dim_{\mathbf{C}} (T_a^{1,0}(X) \cap N_a)$$

is called the holomorphic dimension of X and denoted by hol. dim X. If X is a complex analytic set contained in M, its holomorphic dimension is equal to the (complex) dimension of X.

Kohn's main theorem on subelliptic estimates is the following [13]:

THEOREM 1. *Let Ω be a pseudo-convex domain, $a \in \bar{\partial} \Omega$, and suppose that $\bar{\partial} \Omega$ is smooth and real-analytic in a neighbourhood of a.*

Let $q > 0$. Then, the $\bar{\partial}$-Neumann problem is subelliptic at a for forms of type (p, q) if the following condition is satisfied:

There is a neighbourhood U of a such that $\partial \Omega \cap U$ contains no germ of a real-analytic set whose holomorphic dimension is $\geqslant q$.

[Theorems 4 and 5 below, due to Diederich-Fornaess [4], imply that this condition is satisfied for any $q > 0$, if $\partial \Omega$ is smooth and real-analytic everywhere.]

We turn now to the problem of finding Stein neighbourhoods of $\overline{\Omega}$.

Let $S \subset \partial \Omega$ be the set of points at which Ω is not strongly pseudo-convex.

We say that the pseudo-convex domain Ω is *regular*, if there exist smooth, locally closed submanifolds $V_1, \ldots, V_r$ of $\partial \Omega$ such that

1°. V_k is contained in $\partial \Omega - \bigcup_{l<k} V_l$ as a closed subset.

2°. $S \subset \bigcup_{k \leqq r} V_k$.

3°. The Levi form of $\partial \Omega$, restricted to $T_a^{1,0}(V_k)$, is positive definite for all $a \in V_k$, all $k = 1, \ldots, r$.

The theorems of Diederich-Fornaess [4] can be stated as follows.

THEOREM 2. *If Ω is regular, then $\overline{\Omega}$ has a fundamental system of pseudo-convex neighbourhoods.*

THEOREM 3. *If* Ω *has a smooth real-analytic boundary, then* Ω *is regular.*

The essential ingredients in the proofs of these two theorems are contained in the next two [4].

THEOREM 4. *Let* Ω *be a pseudo-convex domain with a smooth boundary, and let* $a \in \partial \Omega$. *Let* U *be a neighbourhood of* a *in* $\mathbf{C}^n$ *such that* $\partial \Omega \cap U$ *is real-analytic.*

Let $q > 0$, *and suppose that* $\partial \Omega \cap U$ *contains the germ of a real-analytic set whose holomorphic dimension is* $\geqslant q$. *Then* $\partial \Omega \cap U$ *contains the germ of a* complex *analytic set of dimension* $\geqslant q$.

THEOREM 5. *Let* X *be a compact, real-analytic set in* $\mathbf{C}^n$. *Then* X *does not contain the germ of any complex analytic set of dimension* >0.

Putting these results together, one obtains the following theorem.

THEOREM 6. *Let* Ω *be a bounded pseudo-convex domain in* $\mathbf{C}^n$ *with a smooth, real-analytic boundary. Then*

a) $\overline{\Omega}$ *has a fundamental system of neighbourhoods that are pseudo-convex, hence Stein.*

b) *For any* $a \in \partial \Omega$, *and any* $q > 0$, *the* $\bar{\partial}$*-Neumann problem is subelliptic at* a *for forms of type* (p, q).

These results and techniques are being very actively pursued at present. Many problems which looked inaccessible until recently have been solved, at least in important special cases. For instance, the Mergelyan theorem for $\overline{\Omega}$ has seen significant progress (see e.g. [9]). So has the question of global defining equations for the boundary of a pseudo-convex domain ([3]). Finally, a beginning has been made in the study of domains whose boundaries do contain complex analytic sets of positive dimension ([2]).

REFERENCES

[1] ANDREOTTI A. and R. NARASIMHAN. Okas' Heftungstemma and the Levi problem for complex spaces. *Trans. Amer. Math. Soc. 76* (1962), pp. 499-509.

[2] BEDFORD, E. and J. E. FORNAESS. Domains with Pseudoconvex Neighbourhood systems. *To Appear.*