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# REMARKS ON THE UNIVERSAL TEICHMÜLLER SPACE[1]

by F. W. GEHRING[2]

## 1. INTRODUCTION

Suppose that $D$ is a simply connected domain of hyperbolic type in the extended complex plane $\overline{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$. Then the hyperbolic or noneuclidean metric $\rho_D$ in $D$ is given by

$$\rho_D(z) = (1 - |g(z)|^2)^{-1} \, |g'(z)| \, ,$$

where $g$ is any conformal mapping of $D$ onto the unit disk $\{z : |z| < 1\}$. For each function $\varphi$ defined in $D$ we introduce the norm

$$\| \varphi \|_D = \sup_{z \in D} \, |\varphi(z)| \, \rho_D(z)^{-2} \, .$$

Next for each function $f$ which is meromorphic and locally univalent in $D$ we let $S_f$ denote the Schwarzian derivative of $f$. At finite points of $D$ which are not poles of $f$, $S_f$ is given by

$$S_f = \left( \frac{f''}{f'} \right)' - \frac{1}{2} \left( \frac{f''}{f'} \right)^2 ,$$

and the definition is extended to $\infty$ and the poles of $f$ by means of inversion.

Now let $L$ denote the lower half plane $\{z = x + iy : y < 0\}$ and let $B_2 = B_2(L, 1)$ denote the complex Banach space of functions $\varphi$ analytic in $L$ with the norm

$$\| \varphi \| = \| \varphi \|_L = \sup_{z \in L} \, 4y^2 \, |\varphi(z)| < \infty \, .$$

Next let $S$ denote the family of functions $\varphi = S_g$ where $g$ is conformal in $L$, and let $T = T(1)$ denote the subfamily of those $\varphi = S_g$ where $g$ has a quasiconformal extension to $\overline{\mathbf{C}}$. From [6] it follows that $\| \varphi \| \leqslant 6$ for all $\varphi \in S$, and hence that

$$T \subset S \subset B_2 \, . \tag{1}$$

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The set $T$ is called the universal Teichmüller space. An important result due to Ahlfors and Bers shows that each Teichmüller space of a Riemann surface $R$ or of a Fuchsian group $G$ has a canonical embedding in the space $T$. See, for example, [3].

It is natural to ask if there exist relations, other than (1), between $S$ and $T$ as subsets of $B_2$. Compactness results for conformal mappings show that $S$ is closed in $B_2$. Hence Bers asked in [2] and [3] if one can characterize $S$ in terms of $T$ as follows.

QUESTION. *Is $S$ the closure of $T$?*

We shall answer this question in the negative by sketching a proof for the following result.

THEOREM 1. *There exists a $\varphi$ in $S$ which does not lie in the closure of $T$.*

On the other hand, we have the following characterization of $T$ in terms of $S$. See [4].

THEOREM 2. *$T$ is the interior of $S$.*

## 2. REFORMULATIONS IN THE PLANE

A set $E \subset \overline{\mathbf{C}}$ is said to be a *quasiconformal circle* if there exists a quasiconformal mapping $f$ defined in $\overline{\mathbf{C}}$ which maps the unit circle $\{z: |z| = 1\}$ onto $E$.

Theorems 1 and 2 are then respectively equivalent to the following two results on plane domains $D$.

THEOREM 3. *There exists a simply connected domain $D$ and a positive constant $\delta$ such that $f(D)$ is not bounded by a quasiconformal circle whenever $f$ is conformal in $D$ with $\| S_f \|_D \leqslant \delta$.*

THEOREM 4. *A simply connected domain $D$ is bounded by a quasiconformal circle if and only if there exists a positive constant $\delta$ such that $f$ is univalent in $D$ whenever $f$ is meromorphic in $D$ with $\| S_f \|_D \leqslant \delta$.*

We give an argument to show the equivalence of Theorems 1 and 3. Suppose first that Theorem 1 holds. Then there exists a $\varphi \in S$ and a $\delta > 0$ such that $\| \psi - \varphi \| > \delta$ for all $\psi \in T$. Choose $g$ conformal in $L$ with $S_g = \varphi$, let $D = g(L)$ and suppose that $f$ is conformal in $D$ with $\| S_f \|_D \leqslant \delta$. Then $h = f \circ g$ is conformal in $L$,

$$S_h = (S_f \circ g)(g')^2 + S_g \tag{2}$$

by the composition law for the Schwarzian derivative, and hence $\psi = S_h \in S$ with

$$\| \psi - \varphi \| = \| S_h - S_g \|_L = \| S_f \|_D \leqslant \delta .$$

Thus $\psi \notin T$, $h$ does not have a quasiconformal extension to $\overline{\mathbf{C}}$, and $\partial f(D) = \partial h(L)$ is not a quasiconformal circle. Hence Theorem 3 holds.

Suppose next that Theorem 3 holds, let $\varphi = S_g$ where $g$ is any conformal mapping of $L$ onto $D$, and choose any $\psi \in S$ with $\| \psi - \varphi \| \leqslant \delta$. Then $\psi = S_h$ where $h$ is conformal in $L$, $f = h \circ g^{-1}$ is conformal in $D$ and from (2) we obtain

$$\| S_f \|_D = \| S_h - S_g \|_L = \| \psi - \varphi \| \leqslant \delta .$$

Hence $\partial h(L) = \partial f(D)$ is not a quasiconformal circle, $h$ does not have a quasiconformal extension to $\overline{\mathbf{C}}$ and $\psi \notin T$. Thus the distance from $\varphi$ to $T$ is at least $\delta$ and Theorem 1 holds.

A simple modification of the above argument yields the equivalence of Theorems 2 and 4.

Theorems 1 and 3 are immediate consequences of the following result.

THEOREM 5. *There exists a simply connected domain $D$ and a positive constant $\delta$ such that $f(D)$ is not a Jordan domain whenever $f$ is conformal in $D$ with $\| S_f \|_D \leqslant \delta$.*

## 3. SPIRALS

The proof of Theorem 5 is based on two results for a class of spirals.

DEFINITION. *We say that an open arc $\alpha$ in $\mathbf{C}$ is a b-spiral from $z_1$ onto $z_2$ if $\alpha$ has the representation*

$$z = (z_1 - z_2) \, r(t) \, e^{it} + z_2 , \quad 0 < t < \infty ,$$

*where $r(t)$ is positive and continuous with*

$$\lim_{t\to 0} r(t) = 1, \quad \lim_{t\to\infty} r(t) = 0,$$

*and where* $r(t_1) \leqslant b\, r(t_2)$ *for all* $t_1, t_2$ *with* $|t_1 - t_2| \leqslant 2\pi$.

When $a$ is a positive constant, the arc

$$\alpha = \{z = e^{(-a+i)t} : 0 < t < \infty\}$$

is an $e^{2\pi a}$-spiral from 1 onto 0. Moreover,

$$k(z)\,|z| = c, \quad \frac{dk}{ds}(z)\,|z|^2 = d \tag{3}$$

for all $z \in \alpha$, where $c$ and $d$ are positive constants with $d = ac^2$, and where $k$ and $s$ denote the curvature and arclength of $\alpha$.

The first result we need shows that a curvature condition, similar to (3), is sufficient to guarantee that an open arc is a $b$-spiral.

LEMMA 1. *Suppose that* $\alpha$ *is an analytic open arc with* 1 *and* 0 *as endpoints, and suppose that*

$$c_1 \leqslant k(z)\,|z| \leqslant c_2, \quad d_1 \leqslant \frac{dk}{ds}(z)\,|z|^2 \leqslant d_2 \tag{4}$$

*for all* $z \in \alpha$, *where* $c_1, c_2, d_1, d_2$ *are positive constants with* $4\pi d_2 < c_1^2$. *Then* $\alpha$ *is a rectifiable* $b$*-spiral from* 1 *onto* 0 *where*

$$b = \frac{c_1\, c_2}{c_1^{\,2} - 4\pi\, d_2}.$$

The second result we require implies that when $b$ is near 1, the points onto which two disjoint $b$-spirals converge either coincide or are separated by a distance greater than $\frac{1}{2b^2}$ times the diameter of the smaller spiral.

LEMMA 2. *Suppose that* $\alpha$ *and* $\beta$ *are disjoint* $b$*-spirals from* $z_1$ *onto* $z_2$ *and from* $w_1$ *onto* $w_2$, *respectively. If* $b \in (1, 2)$, *then either* $z_2 = w_2$ *or*

$$|z_2 - w_2| > \frac{1}{b} \min(|z_1 - z_2|, |w_1 - w_2|).$$

## 4. Outline of the proof of Theorem 5

Fix $a \in \left(0, \frac{1}{8\pi}\right)$ and let $D = \overline{\mathbf{C}} - \gamma$, where

$$\gamma = \{z = \pm i e^{(-a+i)t} : 0 \leqslant t < \infty\} \cup \{0\}.$$

Then $D$ is a simply connected domain which contains the disjoint $e^{2\pi a}$-spirals

$$\alpha = \{z = e^{(-a+i)t} : 0 < t < \infty\}, \quad \beta = \{z : -z \in \alpha\}.$$

Next let $f$ denote any conformal mapping of $D$ which fixes the points $1, -1, \infty$. To complete the proof of Theorem 5 it is sufficient to show that there exists a positive constant $\delta = \delta(a)$ such that $f(D)$ is not a Jordan domain whenever $\| S_f \|_D \leqslant \delta$. This is done in three steps.

First using Lemma 1 and a normal family type argument, we can prove that there exists a $\delta_1 = \delta_1(a) > 0$ with the following property. If $\| S_f \|_D \leqslant \delta_1$, then $f(\alpha)$ and $f(\beta)$ are $b$-spirals from 1 onto $z_2$ and from $-1$ onto $w_2$, respectively, where $b \in (1, 2)$. The points $z_2, w_2$ are the values which $f(z)$ approaches as $z \to 0$ from opposite sides of $\partial D = \gamma$.

Next theorems on quasiconformal mappings due to Ahlfors [1] and Teichmüller [8] imply the existence of a positive constant $\delta_2 = \delta_2(a) \leqslant \delta_1$ such that $| z_2 | \leqslant \frac{1}{5}$ and $| w_2 | \leqslant \frac{1}{5}$ whenever $\| S_f \|_D \leqslant \delta_2$.

Finally set $\delta = \delta_2$. If $\| S_f \|_D \leqslant \delta$, then

$$| z_2 - w_2 | \leqslant \frac{2}{5} < \frac{4}{5b} \leqslant \frac{1}{b} \min(|1 - z_2|, |-1 - w_2|),$$

Lemma 2 implies that $z_2 = w_2$ and hence $f(D)$ is not a Jordan domain.

A complete proof for Theorem 5 is given in [5].

## 5. Concluding remarks

We have obtained Theorems 1 and 3 from the stronger conclusion in Theorem 5. We conclude by stating a result for multiply connected domains which implies Theorems 2 and 4.

Given a function $\varphi$ defined in an arbitrary proper subdomain $D$ of $\mathbf{C}$, we introduce the norm

$$\| \varphi \|_D^* = \sup_{z \in D} | \varphi(z) | \operatorname{dist}(z, \partial D)^2 .$$

When $D$ is simply connected, classical estimates due to Koebe and Schwarz imply that

$$\frac{1}{4} \operatorname{dist}(z, \partial D)^{-1} \leqslant \rho_D(z) \leqslant \operatorname{dist}(z, \partial D)^{-1}$$

for $z \in D$, and hence that

$$\| \varphi \|_D^* \leqslant \| \varphi \|_D \leqslant 16 \| \varphi \|_D^* .$$

Theorem 6 in [4] and a recent result due to B. Osgood [7] yield the following extension of Theorem 4.

THEOREM 6. *A finitely connected proper subdomain $D$ of* **C** *is bounded by quasiconformal circles or points if and only if there exists a positive constant $\delta$ such that $f$ is univalent in $D$ whenever $f$ is meromorphic in $D$ with $\| S_f \|_D^* \leqslant \delta$.*

## REFERENCES


[1] AHLFORS, L. V. Quasiconformal reflections. *Acta Math. 109* (1963), pp. 291-301.

[2] BERS, L. On boundaries of Teichmüller spaces and on kleinian groups I. *Ann. of Math. 91* (1970), pp. 570-600.

[3] —— Uniformization, moduli, and kleinian groups. *Bull. London Math. Soc. 4* (1972), pp. 257-300.

[4] GEHRING, F. W. Univalent functions and the Schwarzian derivative. *Comm. Math. Helv. 52* (1977), pp. 561-572.

[5] —— Spirals and the universal Teichmüller space. *Acta Math. 141* (1978) (to appear).

[6] KRAUS, W. Über den Zusammenhang einiger Charakteristiken eines einfach zusammenhängened Bereiches mit der Kreisabbildung. *Mitt. Math. Sem. Giessen 21* (1932), pp. 1-28.

[7] OSGOOD, B. Univalence criteria in multiply connected domains. *(To appear)*.

[8] TEICHMÜLLER, O. Extremale quasikonforme Abbildungen und quadratische Differentiale. *Abh. Preuss. Akad. Wiss. 22* (1940), pp. 1-197.


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