1. Introduction

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REMARKS ON THE UNIVERSAL TEICHMÜLLER SPACE[1]

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1. INTRODUCTION

Suppose that D is a simply connected domain of hyperbolic type in the extended complex plane $\overline{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$. Then the hyperbolic or noneuclidean metric ρ_D in D is given by

$$\rho_D(z) = (1 - |g(z)|^2)^{-1} \, |g'(z)| \, ,$$

where g is any conformal mapping of D onto the unit disk $\{z : |z| < 1\}$. For each function φ defined in D we introduce the norm

$$\| \varphi \|_D = \sup_{z \in D} |\varphi(z)| \, \rho_D(z)^{-2} \, .$$

Next for each function f which is meromorphic and locally univalent in D we let S_f denote the Schwarzian derivative of f. At finite points of D which are not poles of f, S_f is given by

$$S_f = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 ,$$

and the definition is extended to ∞ and the poles of f by means of inversion.

Now let L denote the lower half plane $\{z = x + iy : y < 0\}$ and let $B_2 = B_2(L, 1)$ denote the complex Banach space of functions φ analytic in L with the norm

$$\| \varphi \| = \| \varphi \|_L = \sup_{z \in L} 4y^2 \, |\varphi(z)| < \infty \, .$$

Next let S denote the family of functions $\varphi = S_g$ where g is conformal in L, and let $T = T(1)$ denote the subfamily of those $\varphi = S_g$ where g has a quasiconformal extension to $\overline{\mathbf{C}}$. From [6] it follows that $\| \varphi \| \leqslant 6$ for all $\varphi \in S$, and hence that

$$T \subset S \subset B_2 \, . \tag{1}$$

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The set T is called the universal Teichmüller space. An important result due to Ahlfors and Bers shows that each Teichmüller space of a Riemann surface R or of a Fuchsian group G has a canonical embedding in the space T. See, for example, [3].

It is natural to ask if there exist relations, other than (1), between S and T as subsets of B_2. Compactness results for conformal mappings show that S is closed in B_2. Hence Bers asked in [2] and [3] if one can characterize S in terms of T as follows.

QUESTION. *Is S the closure of T?*

We shall answer this question in the negative by sketching a proof for the following result.

THEOREM 1. *There exists a φ in S which does not lie in the closure of T.*

On the other hand, we have the following characterization of T in terms of S. See [4].

THEOREM 2. *T is the interior of S.*

2. REFORMULATIONS IN THE PLANE

A set $E \subset \overline{\mathbf{C}}$ is said to be a *quasiconformal circle* if there exists a quasiconformal mapping f defined in $\overline{\mathbf{C}}$ which maps the unit circle $\{z: |z| = 1\}$ onto E.

Theorems 1 and 2 are then respectively equivalent to the following two results on plane domains D.

THEOREM 3. *There exists a simply connected domain D and a positive constant δ such that $f(D)$ is not bounded by a quasiconformal circle whenever f is conformal in D with $\| S_f \|_D \leqslant \delta$.*

THEOREM 4. *A simply connected domain D is bounded by a quasiconformal circle if and only if there exists a positive constant δ such that f is univalent in D whenever f is meromorphic in D with $\| S_f \|_D \leqslant \delta$.*