5. CONCLUDING REMARKS

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4. Outline of the proof of Theorem 5

Fix $a \in \left(0, \frac{1}{8\pi}\right)$ and let $D = \overline{\mathbf{C}} - \gamma$, where

$$\gamma = \{z = \pm i e^{(-a+i)t} : 0 \leqslant t < \infty\} \cup \{0\}.$$

Then D is a simply connected domain which contains the disjoint $e^{2\pi a}$-spirals

$$\alpha = \{z = e^{(-a+i)t} : 0 < t < \infty\}, \quad \beta = \{z : -z \in \alpha\}.$$

Next let f denote any conformal mapping of D which fixes the points $1, -1, \infty$. To complete the proof of Theorem 5 it is sufficient to show that there exists a positive constant $\delta = \delta(a)$ such that $f(D)$ is not a Jordan domain whenever $\| S_f \|_D \leqslant \delta$. This is done in three steps.

First using Lemma 1 and a normal family type argument, we can prove that there exists a $\delta_1 = \delta_1(a) > 0$ with the following property. If $\| S_f \|_D \leqslant \delta_1$, then $f(\alpha)$ and $f(\beta)$ are b-spirals from 1 onto z_2 and from -1 onto w_2, respectively, where $b \in (1, 2)$. The points z_2, w_2 are the values which $f(z)$ approaches as $z \to 0$ from opposite sides of $\partial D = \gamma$.

Next theorems on quasiconformal mappings due to Ahlfors [1] and Teichmüller [8] imply the existence of a positive constant $\delta_2 = \delta_2(a) \leqslant \delta_1$ such that $| z_2 | \leqslant \frac{1}{5}$ and $| w_2 | \leqslant \frac{1}{5}$ whenever $\| S_f \|_D \leqslant \delta_2$.

Finally set $\delta = \delta_2$. If $\| S_f \|_D \leqslant \delta$, then

$$| z_2 - w_2 | \leqslant \frac{2}{5} < \frac{4}{5b} \leqslant \frac{1}{b} \min\left(|1 - z_2|, |-1 - w_2|\right),$$

Lemma 2 implies that $z_2 = w_2$ and hence $f(D)$ is not a Jordan domain.

A complete proof for Theorem 5 is given in [5].

5. Concluding remarks

We have obtained Theorems 1 and 3 from the stronger conclusion in Theorem 5. We conclude by stating a result for multiply connected domains which implies Theorems 2 and 4.

Given a function φ defined in an arbitrary proper subdomain D of $\mathbf{C}$, we introduce the norm

$$\| \varphi \|_D^* = \sup_{z \in D} | \varphi(z) | \operatorname{dist}(z, \partial D)^2 .$$

When D is simply connected, classical estimates due to Koebe and Schwarz imply that

$$\frac{1}{4} \operatorname{dist}(z, \partial D)^{-1} \leqslant \rho_D(z) \leqslant \operatorname{dist}(z, \partial D)^{-1}$$

for $z \in D$, and hence that

$$\| \varphi \|_D^* \leqslant \| \varphi \|_D \leqslant 16 \| \varphi \|_D^* .$$

Theorem 6 in [4] and a recent result due to B. Osgood [7] yield the following extension of Theorem 4.

THEOREM 6. *A finitely connected proper subdomain* D *of* $\mathbf{C}$ *is bounded by quasiconformal circles or points if and only if there exists a positive constant* δ *such that* f *is univalent in* D *whenever* f *is meromorphic in* D *with* $\| S_f \|_D^* \leqslant \delta$.

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