1. Dimension of D-Modules

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **24 (1978)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **13.09.2024**

ALGEBRAIC ASPECTS OF THE THEORY OF PARTIAL DIFFERENTIAL EQUATIONS [1]

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This is intended to be a report on some recent work in the theory of linear partial differential equations with analytic coefficients. The point of view is here to focus attention, not mainly on the solutions of the equations, but on the structure of the system of equations itself and more precisely on the module over the ring of differential operators defined by this system. The result is a kind of non-commutative algebraic (or better, analytic) geometry, which is rapidly growing now; as one will see from the references, the main contributor is M. Kashiwara.

In this report, we will limit ourselves to the **C**-analytic case, and therefore omit the applications to analysis, which require, of course, looking at the **R**-analytic case. We will mention also very briefly the fundamental tool of "microlocalization", or localization in the cotangent space; but the reader should not forget that this localization plays a fundamental role in the theory, and in many proofs (f.i. in the proof of the "involutiveness of characteristics"), and should therefore be much more fully developed in a systematic exposition.

1. DIMENSION OF $\mathscr{D}$-MODULES

Let X be a **C**-analytic manifold, and n its dimension. We denote by $\mathscr{O}_X$ (or $\mathscr{O}$) the sheaf of holomorphic functions on X, and by $\mathscr{D}_X$ (or $\mathscr{D}$) the sheaf of linear differential operators on X with coefficients in $\mathscr{O}$; we denote by $\mathscr{D}_k$ the subsheaf of operators of degree $\leqslant k$, and by $\mathscr{D}'_m$ the subsheaf of $\mathscr{D}_m$ of operators without constant term; as it is well-known, $\mathscr{D}_0$ can be identified with $\mathscr{O}$, and $\mathscr{D}'_1$ with the sheaf of vector fields on X; moreover, if we denote by $T^*X \overset{\pi}{\to} X$ the cotangent bundle of X, then $\operatorname{gr} \mathscr{D} = \oplus \mathscr{D}_m / \mathscr{D}_{m-1}$ is naturally isomorphic to the subsheaf of $\pi_* (\mathscr{O}_{T^*X})$ of functions "polynomial with respect to the variables of the fibre". More explicitly, if U is an open set of X admitting local coordinates $x_1, \ldots, x_n$, then

[1] Communicated to an International Symposium on Analysis, held in honour of Professor Albert Pfluger, ETH Zürich, 1978.

$\Gamma(U, \mathrm{gr}\,\mathscr{D}) = \Gamma(U, \mathscr{O})[\xi_1, ..., \xi_n]$, where ξ_i is the image of $\partial_i = \dfrac{\partial}{\partial x_i}$ in gr $\mathscr{D}$. From this results easily that $\mathscr{D}$ as sheaf of rings is left and right coherent.

Let M be a "system of p.d.e." on X, i.e. a coherent left $\mathscr{D}$-Module; a filtration on M is an increasing sequence of sub $\mathscr{O}$-Modules M_k verifying $M = \cup M_k$, $D_l M_k \subset M_{k+l}$ for all l, k; the filtration is called "good" if the two following conditions are satisfied

(GF 1) For every k, M_k is coherent over $\mathscr{O}$.

(GF 2) There exists $k_0 \in \mathbf{N}$ such that $D_l M_{k_0} = M_{k_0+l}$, for every $l \in \mathbf{N}$.

Locally, any coherent left $\mathscr{D}$-Module M admits a good filtration; now, one defines the "characteristic variety" of M, char M as follows: choose locally a good filtration $\{ M_k \}$, and consider gr M; as gr $\mathscr{D}$-Module, it is coherent, and therefore its support V in T^*X is well-defined, as an analytic subset of T^*X, relatively algebraic and homogeneous with respect to π, i.e. with respect to variables of the fiber. Note that gr M could depend on the (good) filtration we have chosen; but it turns out that V is independent of it, as is the multiplicity of gr M at any point of V. By definition, we have $V = \mathrm{char}\, M$, and $\dim_a M = \dim_a V$ $(a \in T^*X)$.

In what follows, we identify X with the 0-section of T^*X; due to the homogeneity of V, if $\pi(a) = x$, one has: $\dim_x M \geqslant \dim_a M$. The first nontrivial result of the theory is the following

THEOREM 1.1 (Bernstein [1] — Björk [2] — Kashiwara [8]). *At any point* $x \in X \cap V$, *one has* $\dim_x M \geqslant n$.

A simple proof is given in [B.L.M.] (it is probably the same as Kashiwara's).

A much deeper result, which was conjectured by Guillemin-Quillen-Sternberg and proved in some special cases by these authors, is due to Sato-Kawai-Kashiwara [S.K.K.]; denote by λ the Liouville form on T^*X (in local coordinates, $\lambda = \Sigma \xi_i dx_i$), and put $\omega = d\lambda$; then ω defines canonically a symplectic structure on T^*X; this structure is related to p.d.e. in the following way: Let P and Q be differential operators of orders p and q respectively, and let $\sigma(P)$, $\sigma(Q)$ be the "symbols" of P and Q, i.e. the images of P and Q in gr $\mathscr{D}$; then $[P, Q] = PQ - QP$ is an operator of order $p + q - 1$ and one has $\sigma[P, Q] = \{ \sigma(P), \sigma(Q) \}$, the Poisson bracket of $\sigma(P)$ and $\sigma(Q)$ with respect to ω. Recall also that an analytic subset V of a symplectic manifold is called "involutive" if the sheaf of functions vanishing on V is stable under Poisson bracket. Then, one has

THEOREM 1.2. *The characteristic variety of any coherent $\mathscr{D}$-Module is involutive.*

The proof given in [S.K.K.] is difficult, and uses "pseudo differential operators of infinite order". Recently, a simpler proof, using only usual pseudo differential operators has been obtained independently by Kashiwara and the author (see a forthcoming lecture in "Séminaire Bourbaki").

As consequences of th. 2, and standard facts of symplectic geometry, one has, for every $a \in V = \text{char}\, M : \dim_a M \geqslant n$; moreover, if $\dim_a V = n$, then V is *lagrangian* near a, i.e., on the smooth part V_s of V, in the neighbourhood of a, one has $\lambda / V_s = 0$. If V is globally of dimension n, and therefore globally lagrangian, then there exists a unique stratification of $U = X \cap V$ into smooth submanifolds U_α such that $V = \bigcup_\alpha \overline{N^* U_\alpha}$, $N^* U_\alpha$ the conormal bundle of U_α in X.

Definition 1.3. A coherent left $\mathscr{D}$-Module M is called "holonomic" (or "maximally overdetermined") *if* $\dim M = n$ (*or, equivalently, if* char M *is lagrangian*).

Using the properties of the multiplicity, one sees the following: if $\dim_x M = n$, i.e. if M is holonomic at x, then M_x is a $\mathscr{D}_x$-module of finite length. Therefore, the holonomic M play in this theory more or less the same role as the closed points and the modules of finite length in algebraic geometry. But their structure is much less known! For instance, in the case $n = 1$, analyzing locally that structure is (essentially) equivalent to classifying differential equations near a singularity, regular or irregular. We mention here that recent progress have been made in that problem; we do not insist on that, which is beyond the scope of this report.

To end this section, a few words on pseudo-differential (or "micro-differential") operators. In the C^∞-case, they are well-known; in the analytic case, they were defined by Boutet de Monvel-Krée [3], and studied systematically in [S.K.K.] in connection with hyperfunctions and microfunctions (the reader who is only interested in microdifferential operators could perhaps read independently chap. II of [S.K.K.], and also a partial exposition in [B.L.M.]). They are defined roughly as follows; let U be an open set in T^*X, and choose local coordinates $x = (x_1, \ldots, x_n)$, $\xi = (\xi, \ldots, \xi_n)$ in U. We define $\mathscr{E}(U) = \{p_j(x, \xi)\}_{j \in \mathbf{Z}}$, $p_j \in \mathscr{O}_{T^*X}(U)$, such that

M.1) p_j is homogeneous of degree j in ξ.

M.2) $\sup |p_j(x, \xi)| \leqslant (-j)!\ R_K^{-j}$ for any $K \subset\subset U$, and $j < 0$.

M.3) $p_j = 0$ for $j >> 0$.

At a point $(x, 0)$, the p_j are homogeneous polynomials of degree j in ξ; therefore, $p_j = 0$ for $j < 0$, and, for $j \geqslant 0$, p_j can be identified with the differential operator $p_j(x_i, \partial_i)$; the formulae for multiplication and change of variables in $\mathscr{E}$ are chosen in order to extend what happens on $\mathscr{D}$. In that way, one get a sheaf $\mathscr{E}$ on T^*X with a filtration $\mathscr{E}_j, j \in \mathbf{Z}$ and a structure of (flat) $\pi^*(\mathscr{D})$-Module. All the properties of $\mathscr{D}$ mentioned before can be extended to $\mathscr{E}$, which is called the sheaf of (convergent) microdifferential operators. Note also the following property: if $p \in \mathscr{E}(U)$ has a symbol $\sigma(p)$ which does not vanish, then p is invertible in $\mathscr{E}(U)$ [3]; from that results easily the following useful property: if M is a coherent $\mathscr{D}$-Module, one has char M = support of $\tilde{M}$ with $\tilde{M} = \mathscr{E} \otimes_{\pi^{-1}\mathscr{D}} \pi^{-1} M$.

A variant of the preceding sheaf with essentially similar properties, is given by the sheaf $\hat{\mathscr{E}}$ of "formal" microdifferential operators (it is defined like $\mathscr{E}$, by just removing M.2). Perhaps this sheaf, or an algebraic counterpart, could have some interest for an *algebraic* theory of $\mathscr{D}$-Modules.

2. General constructions on $\mathscr{D}$ and $\mathscr{E}$-Modules

(2.1) *Canonical transformations.*

This operation is restricted to $\mathscr{E}$-Modules on open sets $U \subset T^*X - X$; this is the analytic counterpart of Maslov's ideas [13] and of the theory of "Fourier integral operators" by Hörmander [7]. Given a homogeneous symplectic diffeomorphism $U \xrightarrow{\varphi} V$, with $U, V \subset T^*X - X$, there exists a (non-unique) isomorphism $\mathscr{E}|U \to \mathscr{E}|V$, which respects the filtrations, and verifies $\sigma\Phi(P) = \sigma(P) \circ \varphi^{-1}$. This is often useful to reduce the support of an $\mathscr{E}$-Module, at least at smooth points, to canonical form. Although this is a very fundamental ingredient of the theory, we will not insist on it here. We just mention that Φ is defined by a suitable holonomic system, whose support (= characteristic variety) is precisely the graph of φ in $U \times V$. For the details, we refer to [S.K.K.]; see also [B.L.M.].