3. FURTHER RESULTS ON HOLONOMIC SYSTEMS

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THEOREM (2.3.1). *If M is holonomic on Y, then the $L_i f^* M$ are holonomic on X* (Kashiwara [11]).

However, one problem here is to find the characteristic varieties (this restriction seems to have no microlocal counterpart). Note also that, in the case of modules over the Weyl algebra, i.e. the algebra of differential operators on $\mathbf{C}^n$ with *polynomial* coefficients, the holonomy of $f^* M$ was proved previously by I.N. Bernstein [1].

The preceding theorem can be stated in a more general context, using local cohomology. If now Z is a closed analytic subset of Y, defined by a coherent $\mathcal{O}_Y$-Ideal J, we define $H^i_{[Z]} M = \varinjlim \operatorname{Ext}^i_{\mathcal{O}_X} (\mathcal{O}_X/J^k, M)$; this is not the "transcendental" local cohomology $H^i_Z M$, but the analytical translation of the local cohomology of schemes; it is easily provided with a structure of $\mathcal{D}_Y$-Module. Now, if $X \subset Y$ is a submanifold, it is easy to prove that one has $L_i f^* M = \bar{f}^* (\underline{H}^{d-i}_{[X]} M)$, with $d = \operatorname{codim}_Y X$. Therefore, theorem (2.3.1) is a special case of the following theorem (same reference):

THEOREM (2.3.3). *If M is holonomic, then the $H^i_{[Z]} M$ are holonomic.*

As an easy consequence, the sheaf of meromorphic sections of a connection with singularities in the sense of Deligne [5] is a holonomic $\mathcal{D}$-Module. In some sense, they are the "general case" of holonomic $\mathcal{D}$-Modules (a problem is to give a meaning to this assertion). In particular, modulo non-singular compactifications, one deduces immediately from that fact the following theorem, proved previously by Björk (unpublished ?): the algebraic de Rham cohomology of an algebraic connection on an affine non-singular $\mathbf{C}$-variety is finite.

3. FURTHER RESULTS ON HOLONOMIC SYSTEMS

First, note that, if M is a coherent left $\mathcal{D}$-Module on X and N any $\mathcal{D}$-Module, then $\operatorname{Hom}_{\mathcal{D}} (M, N)$ can be interpreted as the set of solutions of the system of p.d.e. defined by M, with values in N (for instance, if J is a left coherent sheaf of ideals of $\mathcal{D}$, and $M = \mathcal{D}/J$, then $\operatorname{Hom}_{\mathcal{D}} (M, N)$ is the set of $n \in N$ annihilated by J). For instance, taking $N = \mathcal{O}$, we get the holomorphic solutions of M; on the other hand, we have seen the relation between $\mathcal{R}\operatorname{Hom} (\mathcal{O}, N)$ and the de Rham cohomology of N.

Note also that $\mathcal{O}$ is holonomic as a $\mathcal{D}$-Module (one has $\mathcal{O} = \mathcal{D}1 \simeq \mathcal{D}/\Sigma\mathcal{D} \frac{\partial}{\partial x_i}$ and therefore char $(\mathcal{O}) = X$, the null section). This explains the interest of the following theorem, due again to Kashiwara [9], [11].

THEOREM 3.1. *If M and N are holonomic, then the sheaves* $\mathrm{Ext}^i_{\mathcal{D}}(M, N)$ *are* **C**-*analytically constructible* (i.e. *there exists a* **C**-*analytic stratification of X, such that, on each stratum, the sheaf is locally isomorphic to the constant sheaf* $\mathbf{C}^l$ *for some l*); *in particular, the fibers* $\mathrm{Ext}^i_{\mathcal{D}}(M, N)_x$ *are finite over* **C**.

Another problem is posed by the systematisation and extension to $\mathcal{D}$-Modules of the known theorems on regular connexions [5]. Here, one needs some regularity assumptions (for instance, the algebraic cohomology of a connection on an affine non-singular algebraic variety is the same as the analytic one when the connection is regular at infinity, but not in general). This subject is rapidly developing at the moment, and we will only mention some references:

a) In Kashiwara-Oshima [12], one will find regular $\mathcal{D}$- or $\mathcal{E}$-Modules, defined, and studied at generic points of the characteristic variety.

b) In Mebkhout [14] and Ramis [15], one will find systematic developments of the Grothendieck comparison theorem, in relation with $\mathcal{D}$-Modules and also with the "Cousin complex" of Grothendieck, in the analytical version of Ramis-Ruget [16].

Finally, I mention that, recently, Kashiwara and Kawai have announced an extension of the comparison theorem to any regular holonomic $\mathcal{D}$-Module.

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