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UNIVALENT FUNCTIONS, SCHWARZIAN DERIVATIVES AND QUASICONFORMAL MAPPINGS[1]

by Olli Lehto

1. Introduction

Univalent functions have been a popular topic in complex analysis for over sixty years. It has also been known for a long time that there are interesting connections between univalence and the Schwarzian derivative. More recently, one has discovered in this interplay the important role of quasiconformal mappings which not only provide a tool but, somewhat surprisingly, are intrinsic in the problem of deducing univalence from the behavior of the Schwarzian. In this survey, we shall describe some recent developments in this area.

After defining plane quasiconformal mappings, we briefly discuss quasicircles in Section 3. These curves, introduced by Pfluger [15] in 1960, play a central role in this survey. Section 4 deals with the problem of measuring the deviation of a simply connected domain A from a disc D by means of the Schwarzian derivative of the conformal mapping function $f: A \to D$. The starting point in Section 5 is the remarkable result that in a simply connected domain, a small Schwarzian derivative implies univalence if and only if the boundary of the domain is a quasicircle. The sufficiency of this condition is due to Ahlfors [1], the necessity to Gehring [2]. This result gives rise to considering the universal Teichmüller space, and in this way various explicit estimations for certain domain constants can be derived ([9]).

2. Quasiconformal mappings

2.1 *Module of a curve family.* Roughly speaking, quasiconformal mappings are homeomorphisms under which conformal invariants remain quasi-invariant. A precise definition can be given, for instance, in terms of the module of curve families. Let A be a domain in the plane and Γ a family

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of Jordan arcs or curves lying in A. Consider non-negative Borel functions ρ in A and denote by $P(\Gamma)$ the family of all such functions with the property $\int_\gamma \rho \, |dz| \geqslant 1$ for every locally rectifiable $\gamma \in \Gamma$. The greatest lower bound

$$M(\Gamma) = \inf_{\rho \in P(\Gamma)} \int_A \rho^2$$

is called the module of the family Γ.

A sense-preserving homeomorphism f of A onto another domain of the plane is a *K-quasiconformal mapping* if

$$M(\Gamma)\,/\,K \leqslant M\big(f(\Gamma)\big) \leqslant K\, M(\Gamma) \tag{2.1}$$

for every family Γ whose elements lie in A. The smallest possible K in (2.1) is called the maximal dilatation of f. A sense-preserving homeomorphism is conformal if and only if it is 1-quasiconformal.

2.2 *Beltrami equation.* Another way to characterize quasiconformality is as follows: A sense-preserving diffeomorphism is K-quasiconformal if it takes infinitesimal circles onto infinitesimal ellipses with a ratio of axes $\leqslant K$. A sense-preserving homeomorphism is K-quasiconformal if it is the limit of K-quasiconformal diffeomorphisms in the topology of locally uniform convergence.

A variant of this definition is based on the notion of L^2-derivatives. A continuous function is said to have L^2-derivatives in A if it is absolutely continuous on lines in A and if its partials, which then exist a.e. in A, are locally square integrable. By use of complex derivatives ∂ and $\bar{\partial}$, one more equivalent definition of quasiconformity is the following: A function f is a K-quasiconformal mapping of A if and only if f has L^2-derivatives in A and satisfies a Beltrami equation $\bar{\partial} f = \mu \, \partial f$ a.e. in A, where the function μ, the complex dilatation of f, is bounded in absolute value by $(K-1)\,/\,(K+1)$.

The existence theorem for Beltrami equations says that every function μ which is measurable in A and for which $\| \mu \|_\infty < 1$ agrees a.e. with the complex dilatation of a quasiconformal mapping of A. By the uniqueness theorem, complex dilatation determines a quasiconformal mapping up to conformal transformations.

For more details about the properties of quasiconformal mappings in the plane we refer to [11].

3. Quasicircles

3.1 *Definition.* A Jordan curve is the image of a circle under a homeomorphism of the plane. If the homeomorphism can be taken to be a K-quasiconformal mapping, the Jordan curve is called a *K-quasicircle.*

For a later application, we need the following result.

Lemma 3.1. *A K-quasicircle is the image of the real axis under a quasiconformal mapping of the plane which is conformal in the upper half-plane and K^2-quasiconformal in the lower half-plane.*

Proof: Let C be a K-quasicircle. Then there is a K-quasiconformal mapping w of the plane which carries the real axis onto C. Let μ denote the complex dilatation of w. By the existence theorem for Beltrami equations, there is a quasiconformal self-mapping h of the upper half-plane with complex dilatation μ. If h is extended to the lower half-plane by reflection in the real axis, we obtain a K-quasiconformal mapping of the plane. Then $w \circ h^{-1}$ has the desired properties: by the uniqueness theorem for Beltrami equations, it is conformal in the upper half-plane, and as a composition of two K-quasiconformal mappings it is K^2-quasiconformal in the lower half-plane.

The notion of a quasicircle was introduced by Pfluger [15]; he arrived at these curves, which he called "kreisähnlich", in connection with a sewing problem for Riemann surfaces. Pfluger proved that a quasicircle, while always of zero area, need not be rectifiable. Later, Gehring and Väisälä [4] showed that the Hausdorff dimension of a quasicircle is always < 2 but can take any value λ, $1 \leqslant \lambda < 2$.

3.2 *Geometric characterization.* The first systematic study of quasicircles is Tienari's thesis [16]. His results were soon overshadowed by Ahlfors [1], who gave an amazingly simple geometric characterization of quasicircles: A Jordan curve passing through ∞ is a quasicircle if and only if for any of its three successive finite points z_1, z_2, z_3, the ratio $|z_1 - z_2| : |z_1 - z_3|$ is uniformly bounded.

The condition of Ahlfors can be modified in various ways. Let $U(z, r) = \{w \| w - z | < r\}$ and let $\mathrm{cl}U$ denote the closure of U. A set E of the extended plane is *b-locally connected* if the following two conditions hold for every finite z and every $r > 0$:

1° Any two points of the set $E \cap \mathrm{cl}\, U(z, r)$ can be joined by an arc lying in $E \cap \mathrm{cl}\, U(z, br)$.

2° Any two points of the set $E - U(z, r)$ can be joined by an arc lying in $E - U(z, r/b)$.

The following result has recently been proved by Gehring [2]:

LEMMA 3.2. *Let the set* C *contain at least two points and bound a simply connected domain* A. *If* A *is b-locally connected, then* C *is a* $c(b)$*-quasicircle, where* $c(b)$ *depends only on* b.

3.3 *Quasiconformal reflection.* Let C be a Jordan curve bounding the domains A and B. A sense-reversing K-quasiconformal mapping $\varphi: A \to B$ is a *K-quasiconformal reflection* in C if φ leaves every point of C invariant.

It is not difficult to prove that C admits a quasiconformal reflection if and only if C is a quasicircle. It follows that a quasiconformal mapping $f: A \to B$ between domains A and B bounded by quasicircles can be extended to a quasiconformal mapping of the plane. In fact, if φ and ψ are quasiconformal reflections in the boundaries ∂A and ∂B, such that φ is defined outside A and ψ in B, then $\psi \circ f \circ \varphi$ extends f quasiconformally.

A quasicircle always admits quasiconformal reflections which are continuously differentiable or even real-analytic. For a K-quasicircle passing through ∞, a reflection φ exists such that $|d\varphi(z)| / |dz|$ is bounded by a constant depending only on K.

For more details of the properties of quasicircles we refer to [10].

4. DEVIATION OF A DOMAIN FROM A DISC

4.1 *Schwarzian derivative.* Let f be a locally injective meromorphic function in a simply connected domain A. At finite points of A which are not poles of f, the *Schwarzian derivative* S_f of f is defined by

$$S_f = (f''/f')' - \frac{1}{2}(f''/f')^2 ,$$

and the definition is extended to ∞ and to the poles of f by means of inversion.

The Schwarzian derivative is holomorphic in A. Conversely, every function which is holomorphic in A is the Schwarzian of some f. The Schwarzian vanishes identically if and only if f is a Möbius transformation.

More generally, the Schwarzian determines a function up to a Möbius transformation.

Suppose the boundary of A consists of more than one point; then a conformal mapping h of A onto the unit disc exists. Through h a conformally invariant metric $\rho(z)\,|dz|$ is defined in A, by the rule $\rho(z)\,|dz| = (1-|w|^2)^{-1}\,|dw|$, $w = h(z)$. For functions φ holomorphic in A we introduce the norm

$$\|\varphi\|_A = \sup_{z\in A} |\varphi(z)|\,\rho(z)^{-2}.$$

The Schwarzian obeys the composition rule $S_{f\circ g} = (S_f \circ g)\, g'^2 + S_g$. We note certain of its immediate consequences. First, let f be meromorphic in A and $h: A \to B$ a conformal mapping. Then

$$(4.1)\qquad |S_f(z) - S_h(z)|\,\rho_A(z)^{-2} = |S_{f\circ h^{-1}}(\zeta)|\,\rho_B(\zeta)^{-2}, \quad \zeta = h(z).$$

It follows that $\| S_f - S_h \|_A = \| S_{f\circ h^{-1}} \|_B$. In particular,

$$(4.2)\qquad \| S_h \|_A = \| S_{h^{-1}} \|_B .$$

Secondly, let f and g be meromorphic in A and $h: G \to A$ a conformal mapping. Then

$$(4.3)\qquad \| S_{f\circ h} - S_{g\circ h} \|_G = \| S_f - S_g \|_A .$$

Finally, we remark that the norm of the Schwarzian is completely invariant under Möbius transformations: If f is meromorphic in A and g and h are Möbius transformations, then $\| S_{h\circ f\circ g} \|_{g^{-1}(A)} = \| S_f \|_A$.

4.2 *Constant* σ_1. We associate with the domain A the constant $\sigma_1 = \| S_f \|_A$, where f is a conformal map of A onto a disc. Here a disc means an ordinary disc or a half-plane. The number σ_1 is well defined, and equal to 0 if and only if A itself is a disc. It can be regarded as a measure of how much the domain A differs from a disc.

It is well known that $\sigma_1 \leqslant 6$ (Theorem of Kraus [6]). For the domain $A = \{z \mid 0 < \arg z < k\pi\}$, $1 \leqslant k \leqslant 2$, we have $\sigma_1 = 2(k^2-1)$. If follows that σ_1 can take any value in the closed interval [0, 6].

4.3 *Domains bounded by a quasicircle.* In some cases, information about the boundary of A makes it possible to improve the estimate $\sigma_1 \leqslant 6$.

THEOREM 4.1. *For a domain A bounded by a K-quasicircle,*

$$(4.4)\qquad \sigma_1 \leqslant 6\,\frac{K^2-1}{K^2+1}.$$

Proof: By Lemma 3.1, there exists a K^2-quasiconformal mapping w of the plane whose restriction to the upper half-plane H maps H conformally onto A. For the function $w \mid H$ the Krauss estimate can be improved:

$$\| S_{w|H} \|_H \leqslant 6 \frac{K^2 - 1}{K^2 + 1};$$

for the proof we refer to Kühnau [7], or to [8]. Hence (4.4) follows from (4.2).

4.4 *Domains with bounded boundary rotation.* Let A be a domain bounded by a continuously differentiable Jordan curve. The total variation of the direction angle of the boundary tangent under a complete circuit is called the boundary rotation of A. If the boundary is not so regular, boundary rotation is defined by means of approximations from inside.

Let f be a conformal mapping of the unit disc D onto a domain A with boundary rotation $k\pi$, $2 \leqslant k < \infty$. A real-valued function ψ with the properties

$$\int_0^{2\pi} d\psi(\theta) = 2, \int_0^{2\pi} |d\psi(\theta)| = k,$$

can be associated with f, such that

$$f'(z) = f'(0) \exp \Big(- \int_0^{2\pi} \log (1 - ze^{-i\theta}) \, d\psi(\theta) \Big). \tag{4.5}$$

The domain A is convex if and only if $k = 2$. This is equivalent to ψ being an increasing function. A function f whose derivative admits the representation (4.5) is always univalent if the total variation of ψ is $\leqslant 4$.

Domains with bounded boundary rotation were introduced by Löwner and their basic properties established by Paatero [14].

THEOREM 4.2. *For a domain A with boundary rotation $\leqslant k\pi$, $2 \leqslant k \leqslant 4$,*

$$\sigma_1 \leqslant \frac{2k + 4}{6 - k}. \tag{4.6}$$

The bound is sharp.

Proof: Let $f : D \to A$ be a conformal mapping, z_0 an arbitrary point of D, and h a conformal self-mapping of D, such that $h(0) = z_0$. Since $\rho_D(0) = 1$, it follows from (4.1) that

(4.7) $$| S_f(z_0) | \rho_D(z_0)^{-2} = | S_{f \circ h}(0) | .$$

Hence, (4.6) follows if we prove that $| S_f(0) | \leqslant (2k+4) / (6-k)$. Since we may replace f by the function $z \to cf(ze^{i\varphi})$, c complex, φ real, there is no loss of generality in assuming that $S_f(0) \geqslant 0$ and that $f'(0) = 1$. From the representation formula (4.5) we then deduce that

(4.8) $$S_f(0) = \int_0^{2\pi} \cos 2\theta \, d\psi(\theta) - \frac{1}{2}\left(\int_0^{2\pi} \cos\theta d\psi(\theta)\right)^2 + \frac{1}{2}\left(\int_0^{2\pi} \sin\theta d\psi(\theta)\right)^2 .$$

If $k = 2$, we have $d\psi(\theta) \geqslant 0$. In this case we get the inequality $\sigma_1 \leqslant 2$ for convex domains from (4.8) quite easily, just by use of Schwarz's inequality. Extremal functions can also be determined. These computations have been carried out in [9]. Nehari [13] proved the result $\sigma_1 \leqslant 2$ by means of variational methods.

If $2 < k \leqslant 4$, establishing (4.6) requires a more careful handling of formula (4.8). These computations will be published in a joint paper with O. Tammi.

Matti Lehtinen has let me know that for functions f whose derivative satisfies (4.5) with a ψ whose total variation is $\leqslant k$, $k \geqslant 4$, the sharp upper bound for $\| S_f \|$ is equal to $(k^2 - 4)/2$. The extremal functions are not univalent.

4.5 *Constant* σ_2. The domain constant

$$\sigma_2 = \sup \{ \| S_f \|_A \mid f \text{ univalent in } A \}$$

is in simple relation with σ_1 ([9]):

THEOREM 4.3. *In every domain* A, $\sigma_2 = \sigma_1 + 6$.

Proof: Let f be univalent in A and $h: D \to A$ conformal. By (4.3),

$$\| S_f \|_A = \| S_{f \circ h} - S_f \|_D \leqslant 6 + \| S_h \|_D = 6 + \sigma_1 .$$

In order to derive an estimate in the opposite direction, let an $\varepsilon > 0$ be given. In view of formula (4.7), we can choose $h: D \to A$ so that $| S_h(0) |$

$> \sigma_1 - \varepsilon$. If w is defined by $w(z) = z + e^{i\theta}/z$ and $f = w \circ h^{-1}$, then f is univalent in A and

$$\| S_f \|_A = \| S_w - S_h \|_D \geqslant | S_w(0) - S_h(0) | = | 6e^{i\theta} + S_h(0) | .$$

By choosing φ suitably we obtain $\| S_f \|_A > 6 + \sigma_1 - \varepsilon$.

5. Schwarzian derivative and univalence

5.1 *Constant* σ_3. Let A again be a simply connected domain with more than one boundary point. As a kind of opposite to the constant σ_2 we define

$$\sigma_3 = \sup \{ a \mid \| S_f \| \leqslant a \text{ implies } f \text{ univalent in } A \}.$$

Note that the number $a = 0$ is always in the above set. In this definition, sup can be replaced by max, as can be shown by a standard normal family argument.

Nehari [12] proved that in a disc, the condition $\| S_f \| \leqslant 2$ implies the univalence of f, and Hille [5] showed that the bound 2 is best possible. In other words, $\sigma_3 = 2$ for a disc.

A closer study of σ_3 leads to the universal Teichmüller space and reveals an intrinsic significance of quasiconformal mappings in the theory of univalent functions. The gist is the following result.

Theorem 5.1. *The constant* σ_3 *is positive if and only if* A *is bounded by a quasicircle.*

Proof: The sufficiency of the condition was established by Ahlfors [1] who actually proved more: If A is bounded by a K-quasicircle, there is an $\varepsilon > 0$ depending only on K, such that whenever $\| S_f \|_A < \varepsilon$, then f is univalent and can be continued to a quasiconformal mapping of the plane. In the proof, the extension of the given meromorphic f is explicitly constructed by means of a continuously differentiable quasiconformal reflection φ in ∂A with bounded $| d\varphi | / | dz |$ (cf. 3.3).

The necessity was proved by Gehring [2]. His proof was in two steps. It was first shown, by aid of an example, that if A is not b-locally connected for any b, then $\sigma_3 = 0$. After this, the desired conclusion was drawn from the result we stated above as Lemma 3.2.

5.2 *Universal Teichmüller space.* Henceforth, we assume that the domain A is bounded by a quasicircle. Let $Q(A)$ be the Banach space consisting of all holomorphic functions φ of A with finite norm. We introduce the subsets

$$U(A) = \{\varphi = S_f \mid f \text{ univalent in } A\},$$
$$T(A) = \{S_f \in U(A) \mid f \text{ can be extended to a quasiconformal mapping of the plane}\}.$$

Both sets are well defined. The set $T(A)$ is called the *universal Teichmüller space* of A.

THEOREM 5.2. *The sets* $T(A)$ *and* $U(A)$ *are connected by the relation* $T(A)$ = interior of $U(A)$.

Proof: We first show that $T(A)$ is open. Choose $S_f \in T(A)$, $S_h \in Q(A)$, and set $g = h \circ f^{-1}$. Then g is meromorphic in the domain $f(A)$. Since ∂A is a quasicircle, $\partial f(A)$ is also a quasicircle. By the theorem of Ahlfors cited in the proof of Theorem 5.1, there is an $\varepsilon > 0$ such that if

$$\| S_g \|_{f(A)} < \varepsilon , \tag{5.1}$$

then $S_g \in T(f(A))$. Now, choose h so that $\| S_f - S_h \|_A < \varepsilon$. Then (5.1) holds, and it follows that $S_h = S_{g \circ f} \in T(A)$.

After this it suffices to prove that $\operatorname{int} U(A) \subset T(A)$. Choose $S_f \in \operatorname{int} U(A)$ and then an $\varepsilon > 0$, so that the ball $B = \{\varphi \in Q(A) \mid \| \varphi - S_f \| < \varepsilon\}$ is contained in $U(A)$. Let g be an arbitrary meromorphic function in $f(A)$ for which $\| S_g \|_{f(A)} < \varepsilon$. If $h = g \circ f$, then $\| S_f - S_h \|_A = \| S_g \|_{f(A)} < \varepsilon$. Thus $S_h \in U(A)$. But then also $g = h \circ f^{-1}$ is univalent, and we have proved that σ_3 is positive for the domain $f(A)$. By Theorem 5.1, the boundary $\partial f(A)$ is a quasicircle. Hence, by the remark in 3.3, $S_f \in T(A)$.

COROLLARY 5.1. *If* f *is univalent in* A *and* $\| S_f \|_A < \sigma_3$, *then* f *can be extended to a quasiconformal mapping of the plane.*

Proof: This follows immediately from Theorem 5.2, in view of our previous remark that the closed ball $\{\varphi \in Q(A) \mid \| \varphi \|_A \leqslant \sigma_3\}$ is contained in $U(A)$.

By this Corollary, we have for A,

$$\sigma_3 = \sup \{a \mid \| S_f \|_A < a \text{ implies that } f \text{ is univalent and can be extended to a quasiconformal mapping of the plane}\}.$$

5.3 *New characterization for* σ_3. Theorem 5.2 was proved by Gehring [2] in the case where A is a half-plane. As is seen from the above proof, the generalization for an arbitrary A is immediate. In fact, the sets $Q(A)$, $U(A)$ and $T(A)$ corresponding to different domains A are isomorphic:

LEMMA 5.1. *Let h be a conformal mapping of the upper half-plane H onto A. Then the mapping $h^*: Q(A) \rightarrow Q(H)$, defined by $h^*(S_f) = S_{f \circ h}$, is a bijective isometry. It maps $U(A)$ and $T(A)$ onto $U(H)$ and $T(H)$, respectively.*

Proof: Clearly h^* is well defined and a bijection of $Q(A)$, $U(A)$ and $T(A)$ onto $Q(H)$, $U(H)$ and $T(H)$, respectively. That h^* is an isometry follows from formula (4.3).

The function h^* maps the origin of $Q(A)$ onto the point $S_h \in T(H)$, which has the distance σ_1 from the origin of $Q(H)$. If $B = \{\varphi \in Q(A) \mid \|\varphi\|_A \leqslant \sigma_3\}$, then

$$h^*(B) = \{\psi \in Q(H) \mid \|\psi - S_h\| \leqslant \sigma_3\}.$$

From this and the definition of σ_3 we infer that σ_3 is equal to the distance from the point S_h to the boundary of $U(H)$. The following characterization seems to be more useful:

LEMMA 5.2. *The constant σ_3 of A is equal to the distance of the point S_h to the boundary of $T(H)$.*

Proof: Let d denote the distance function in Q. Since $T(H) \subset U(H)$ we conclude from what we just said above that $\sigma_3 \geqslant d(\{S_h\}, U(H) - T(H))$. On the other hand, it follows from Theorem 5.2 that int $B \subset T(A)$ and hence int $h^*(B) \subset T(H)$. Therefore, $\sigma_3 \leqslant d(\{S_h\}, U(H) - T(H))$.

A standard normal family argument shows that $U(A)$ is a closed subset of $Q(A)$. Therefore, the closure of $T(A)$ is contained in $U(A)$. Gehring [3] showed recently that this inclusion is proper, thus disproving a famous conjecture of Bers.

However, it is true that on every sphere $\|\varphi\| = r$ of $Q(H)$, $2 \leqslant r \leqslant 6$, there are points of $U(H) - T(H)$ which belong to the closure of $T(H)$ ([9]).

5.4 *Estimates for* σ_3. Lemma 5.2 can be used to deriving estimates for σ_3 in terms of σ_1 ([9]). Suppose first that $0 \leqslant \sigma_1 < 2$. Then S_h lies in the ball $\{\varphi \in Q(H) \mid \|\varphi\| < 2\}$ which is a subset of $T(H)$. Since $\|S_h\| = \sigma_1$, we conclude that $d(\{S_h\}, U(H)-T(H)) \geqslant 2 - \sigma_1$. Consequently, by Lemma 5.2,

$$\sigma_3 \geqslant 2 - \sigma_1 \,. \tag{5.2}$$

In order to prove that this inequality is sharp, we consider the point S_w, where w is the restriction to H of a branch of the logarithm. Since the boundary of $w(H)$ is not a quasicircle, $S_w \in U(H) - T(H)$. From $S_w(z) = z^{-2}/2$ it follows that $\| S_w \|_H = 2$. Let h be determined by the condition $S_h = r\, S_w$, $0 < r < 1$, and set $A = h(H)$. From $\| S_h \|_H < 2$ it follows that $S_h \in T(H)$, and so ∂A is a quasicircle. Now

$$\sigma_3 = d(\{S_h\}, U(H) - T(H)) = \| S_w - S_h \| = 2(1-r) = 2 - \sigma_1 \,,$$

showing that (5.2) is sharp.

Suppose that $2 \leqslant \sigma_1 < 6$. We then conclude from the remark at the end of 5.3 that, even though $\sigma_3 > 0$ for each A, we have $\inf \sigma_3 = 0$ for every σ_1.

Similarly, Lemma 5.2 can be used to deriving the upper estimate

$$\sigma_3 \leqslant \min(2, 6 - \sigma_1) \,.$$

(For the details we refer to [9].)

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