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Autor: Lehto, Olli
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3. QUASICIRCLES

3.1 *Definition.* A Jordan curve is the image of a circle under a homeomorphism of the plane. If the homeomorphism can be taken to be a K -quasiconformal mapping, the Jordan curve is called a K -quasicircle.

For a later application, we need the following result.

LEMMA 3.1. *A K -quasicircle is the image of the real axis under a quasiconformal mapping of the plane which is conformal in the upper half-plane and K^2 -quasiconformal in the lower half-plane.*

Proof: Let C be a K -quasicircle. Then there is a K -quasiconformal mapping w of the plane which carries the real axis onto C . Let μ denote the complex dilatation of w . By the existence theorem for Beltrami equations, there is a quasiconformal self-mapping h of the upper half-plane with complex dilatation μ . If h is extended to the lower half-plane by reflection in the real axis, we obtain a K -quasiconformal mapping of the plane. Then $w \circ h^{-1}$ has the desired properties: by the uniqueness theorem for Beltrami equations, it is conformal in the upper half-plane, and as a composition of two K -quasiconformal mappings it is K^2 -quasiconformal in the lower half-plane.

The notion of a quasicircle was introduced by Pfluger [15]; he arrived at these curves, which he called “kreisähnlich”, in connection with a sewing problem for Riemann surfaces. Pfluger proved that a quasicircle, while always of zero area, need not be rectifiable. Later, Gehring and Väisälä [4] showed that the Hausdorff dimension of a quasicircle is always < 2 but can take any value λ , $1 \leq \lambda < 2$.

3.2 *Geometric characterization.* The first systematic study of quasicircles is Tienari's thesis [16]. His results were soon overshadowed by Ahlfors [1], who gave an amazingly simple geometric characterization of quasicircles: A Jordan curve passing through ∞ is a quasicircle if and only if for any of its three successive finite points z_1, z_2, z_3 , the ratio $|z_1 - z_2| : |z_1 - z_3|$ is uniformly bounded.

The condition of Ahlfors can be modified in various ways. Let $U(z, r) = \{w \mid |w - z| < r\}$ and let $\text{cl}U$ denote the closure of U . A set E of the extended plane is *b-locally connected* if the following two conditions hold for every finite z and every $r > 0$:

- 1° Any two points of the set $E \cap \text{cl}U(z, r)$ can be joined by an arc lying in $E \cap \text{cl}U(z, br)$.
- 2° Any two points of the set $E - U(z, r)$ can be joined by an arc lying in $E - U(z, r/b)$.

The following result has recently been proved by Gehring [2]:

LEMMA 3.2. *Let the set C contain at least two points and bound a simply connected domain A . If A is b -locally connected, then C is a $c(b)$ -quasicircle, where $c(b)$ depends only on b .*

3.3 *Quasiconformal reflection.* Let C be a Jordan curve bounding the domains A and B . A sense-reversing K -quasiconformal mapping $\varphi: A \rightarrow B$ is a K -quasiconformal reflection in C if φ leaves every point of C invariant.

It is not difficult to prove that C admits a quasiconformal reflection if and only if C is a quasicircle. It follows that a quasiconformal mapping $f: A \rightarrow B$ between domains A and B bounded by quasicircles can be extended to a quasiconformal mapping of the plane. In fact, if φ and ψ are quasiconformal reflections in the boundaries ∂A and ∂B , such that φ is defined outside A and ψ in B , then $\psi \circ f \circ \varphi$ extends f quasiconformally.

A quasicircle always admits quasiconformal reflections which are continuously differentiable or even real-analytic. For a K -quasicircle passing through ∞ , a reflection φ exists such that $|d\varphi(z)|/|dz|$ is bounded by a constant depending only on K .

For more details of the properties of quasicircles we refer to [10].

4. DEVIATION OF A DOMAIN FROM A DISC

4.1 *Schwarzian derivative.* Let f be a locally injective meromorphic function in a simply connected domain A . At finite points of A which are not poles of f , the *Schwarzian derivative* S_f of f is defined by

$$S_f = (f''/f')' - \frac{1}{2}(f''/f')^2,$$

and the definition is extended to ∞ and to the poles of f by means of inversion.

The Schwarzian derivative is holomorphic in A . Conversely, every function which is holomorphic in A is the Schwarzian of some f . The Schwarzian vanishes identically if and only if f is a Möbius transformation.