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## MISCONCEPTIONS ABOUT $K_X$

by Steven L. KLEIMAN

There are three common misconceptions about the sheaf  $K_X$  of meromorphic functions on a ringed space  $X$ : (1) that  $K_X$  can be defined as the sheaf associated to the presheaf of total fraction rings,

$$(*) \quad U \mapsto \Gamma(U, O_X)_{tot},$$

see [EGA IV<sub>4</sub>, 20.1.3, p. 227] and [1, (3.2), p. 137]; (2) that the stalks  $K_{X,x}$  are equal to the total fraction rings  $(O_{X,x})_{tot}$ , see [EGA IV<sub>4</sub>, 20.1.1 and 20.1.3, pp. 226-7]; and (3) that if  $X$  is a scheme and  $U = \text{Spec}(A)$  is an affine open subscheme, then  $\Gamma(U, K_X)$  is equal to  $A_{tot}$ , or in other words, the presheaf (\*) is a sheaf if  $U$  ranges exclusively over affines, see [3, Def., p. 140]. These misconceptions will be corrected below with some observations and examples.

The presheaf (\*) may fail to exist! Some restriction maps may simply not be defined. For instance, there may be a nonzerodivisor  $t$  in  $\Gamma(X, O_X)$  whose restriction is a zerodivisor in  $\Gamma(U, O_X)$  for some open subset  $U$ . Then the fraction  $1/t$  in  $\Gamma(X, O_X)_{tot}$  has no restriction in  $\Gamma(U, O_X)_{tot}$ .

For example, let  $A$  be a domain with nonzero maximal ideal  $M$ . Let  $P$  denote the projective line over  $A$ , and  $Y$  the (closed) fiber over  $M$ . Set

$$X = \text{Spec}(O_P \oplus O_Y(-1)),$$

where  $O_Y(-1)$  is viewed as an ideal of square zero. We have

$$\Gamma(X, O_X) = \Gamma(P, O_P) \oplus \Gamma(Y, O_Y(-1)) = A.$$

Hence any nonzero element  $t$  of  $M$  is a nonzerodivisor in  $\Gamma(X, O_X)$ . However, for any affine open subset  $U$  of  $X$  containing a point of  $Y$ , the restriction of  $t$  in  $\Gamma(U, O_X)$  is zerodivisor. Indeed,  $O_Y(-1)|_U$  is isomorphic to  $O_Y|_U$ . So  $\Gamma(U, O_Y(-1))$  contains a nonzero element  $s$ , and obviously  $ts = 0$  holds. Note that if  $A$  is taken to be a finitely generated algebra over a field, then  $X$  is an algebraic scheme for which the presheaf (\*) is undefined.

The right way to define  $K_X$  is as the sheaf associated to the following presheaf of rings of fractions:

$$(**) \quad U \mapsto \Gamma(U, \mathcal{O}_X) [S(U)^{-1}],$$

where  $S(U)$  denotes the set of elements of  $\Gamma(U, \mathcal{O}_X)$  whose restrictions are nonzerodivisors in the stalks  $\mathcal{O}_{X,x}$  for all  $x \in X$ . Note that  $S(U)$  is contained in the set of nonzerodivisors in  $\Gamma(U, \mathcal{O}_X)$ . Hence the presheaf  $(**)$  is separated; that is, the natural map from it to  $K_X$  is injective.

The natural map from  $\mathcal{O}_X$  to  $K_X$  is injective because the one to the presheaf  $(**)$  is and the latter is separated (alternatively, and sheaving is exact). Now, let  $f: X \rightarrow Y$  be a flat map, for example, an open embedding. Then  $f$  gives rise naturally to a map,  $f^*: K_Y \rightarrow f^* K_X$ . So  $K_X$  will work out well in a theory of (Cartier) divisors.

If  $X$  is a scheme and  $U = \text{Spec}(A)$  is an affine open subscheme, then  $S(U)$  contains (and so consists of) all nonzerodivisors  $t \in A$ . Indeed, suppose  $t(a/b) = 0$  holds in  $\mathcal{O}_{X,x}$  for some  $x \in U$  with  $a, b \in A$  and  $b(x) \neq 0$ . Then  $tca = 0$  holds in  $A$  for some  $c \in A$  with  $c(x) \neq 0$ . Since  $t$  is a nonzerodivisor,  $ca = 0$  holds in  $A$ . Hence  $a/b = 0$  holds in  $\mathcal{O}_{X,x}$ , q.e.d. Therefore, when  $X$  is a scheme, the presheaf  $(*)$  will be well-defined (and equal to the presheaf  $(**)$ ) if  $U$  ranges exclusively over affines, and the associated sheaf is  $K_X$ .

Fix  $x \in X$  and set  $S_x = \varinjlim \{S(U) \mid U \in x\}$ . We have

$$K_{X,x} = S_x^{-1} \mathcal{O}_{X,x} \subset (\mathcal{O}_{X,x})_{tot}.$$

The inclusion may be proper, even if  $X$  is an affine scheme. For example, let  $B$  be a domain with a nonzero and nonmaximal ideal  $p$  such that  $p$  is the intersection of all the maximal ideals  $M$  containing it. Set

$$X = \text{Spec}(B \oplus (\oplus_{M \supset p} (B/M))).$$

Let  $x \in X$  represent  $p$ . Then  $K_{X,x}$  is equal to  $B_p$ , while  $(\mathcal{O}_{X,x})_{tot}$  is equal to the fraction field of  $B$ .

The presheaf  $(**)$  need not be a sheaf. In fact, there is an affine scheme  $X = \text{Spec}(A)$  such that  $A_{tot}$  is a proper subring of  $\Gamma(X, K_X)$ . To construct  $X$ , fix an algebraically closed ground field  $k$ , and a smooth closed cubic  $E$  in  $\mathbf{P}_k^2$ . Let  $L$  be a line section of  $E$ , and  $P \in E$  a  $k$ -point such that the divisor  $(3P - L)$  has infinite order; for example, take a  $P$  whose coordinates are transcendental over the field of definition of  $E$ . Let  $C$  be a cone in  $\mathbf{A}_k^3$  pro-

jecting  $E$ , and denote by  $G$  the generator over  $P$ . Take planes  $H_1$  and  $H_2$  through the vertex with no generator in common and neither one containing  $G$ . Take a plane  $H_3$  not containing the vertex, parallel to  $G$ , but not parallel to any generator on  $H_1$ . Denote by  $U$  the set of closed points  $C$  off  $(G \cup H_1) - H_3$ . Set

$$X = \text{Spec} (O_C \oplus (\oplus_{Q \in U} k(Q))).$$

Finally, let  $f$  be a function on  $E$  with a single pole of order 2 at  $P$ , and view  $f$  as a global section of the sheaf  $K_C \oplus (\oplus k(Q))$ , which contains  $K_X$ .

Then  $f$  is in  $\Gamma(X, K_X)$  because  $X$  is covered by the three affine open subsets  $V_i = X - H_i$  and  $f$  is easily seen to be in each  $\Gamma(V_i, O_X)_{tot}$ . However,  $f$  is not in  $\Gamma(X, O_X)_{tot}$ . Indeed, suppose  $f$  is equal to  $r/s$  with  $r, s \in \Gamma(X, O_X)$ . Write  $s = t + \tau$  with  $t \in \Gamma(X, O_C)$  and  $\tau \in \Gamma(X, \oplus k(Q))$ . Then  $t$  is the restriction to  $C$  of a polynomial function on  $\mathbf{A}_k^3$ . So the zero locus  $Z(t)$  is a hypersurface section of  $C$ . Hence, by the choice of  $P \in E$ , there must be a component  $Z$  of  $Z(t)$  different from  $G$ . By construction,  $U$  must contain a point  $Q$  of  $Z$ . Therefore  $t$  is a zerodivisor in  $\Gamma(X, O_X)$ , so  $s$  is also, q.e.d.

Lastly, consider two common cases: (a)  $X$  is a locally noetherian scheme, and (b)  $X$  is a reduced scheme whose set of irreducible components is locally finite. In both cases, Assertions (2) and (3) at the beginning are valid, and  $K_X$  is given by the formula,

$$K_X = j_* (O_X | \text{Ass}(X)),$$

where  $\text{Ass}(X)$  denotes the set of points  $x \in X$  where the maximal ideal of  $O_{X,x}$  is associated to 0, and  $j$  denotes the inclusion map of  $\text{Ass}(X)$  into  $X$ . These statements are easily verified using the ideas in the proof of [EGA IV<sub>4</sub>, 20.2.11, pp. 234-5]. (For a different slant on Case (a), see [4, Lecture 9, 1°, pp. 61-2].)

In Case (b),  $K_X$  is quasi-coherent. However, in Case (a) it need not be. For example, let  $A = k[s, t]$  be the polynomial ring over a field  $k$ , and set

$$X = \text{Spec} (A \oplus A/(s, t)).$$

Though injective, the natural map from  $A_{tot} [1/s]$  into  $A [1/s]_{tot}$  is not surjective; the image omits  $1/t$ . So  $K_X$  is not quasi-coherent.

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