III. Recent progress on existence problems

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The question arises to what extent this is a satisfactory solution to the problem: among all surfaces with a given boundary, find one with least area. This gives rise to the regularity problem, that is showing that the solutions thus found are indeed manifolds or manifolds outside a small singular set.

The theory of normal and integral currents, as developed by Federer and Fleming [F-F] in their fundamental paper of 1960, is essentially a theory of chains with real or integer coefficients, which has both all reasonable properties of algebraic topology and which at the same time yields reasonable spaces for the purpose of the calculus of variations. However, it is not entirely suitable to study the actual soap films which come out in physical experiments. One difficulty is because of orientation; for example, a Möbius band usually arises as a soap film off a wire which approaches a doubly covered circle. This difficulty can be overcome by working with currents with finite abelian coefficient group, for example with mod 2 coefficients. On the other hand, it became increasingly clear that in order to get a theory suitable for describing physical experiments one had to work in a more set theoretic fashion and give up the useful notion of boundary operator. A convenient theory is the theory of varifolds by Almgren. A varifold is simply a Radon measure on the Grassmann bundle of the space. An appropriate notion of rectifiable and integral varifold is developed, and analogs of the closure, compactness and approximation theorems can be obtained. There are some important differences however and it turns out that currents and varifolds complement each other in several respects.

We end this section by referring to Federer, [FH 1] Ch. IV and Almgren, [AF 1] for precise definitions and proofs of the basic properties of currents and varifolds. All these concepts can be extended to ambient spaces different from euclidean space and in particular to Riemannian manifolds.

III. Recent progress on existence problems

One of the great successes of the Calculus of Variations in the Large has been the proof of existence of closed geodesics on smooth compact Riemannian manifolds. The following striking result is a two dimensional extension.

THEOREM 1 (J. Pitts [P 1, 2]. *Every smooth, compact, three dimensional Riemannian manifold without boundary contains a non-empty, closed, imbedded, two dimensional, minimal submanifold without boundary.*

Suppose M is a smooth compact Riemannian manifold of dimension n and let $0 \leqq k \leqq n$. It is still an open question whether M always contains a regular closed minimal submanifold of dimension k. If $k = 1$, this is the problem of existence of simple closed geodesics, which can be treated using Morse theory. If $k \geqq 2$, existing results required suitable assumptions on M. For example, if $k = n - 1$ and $H_{n-1}(M, \mathbf{Z}) \neq 0$ one can find a closed current T with $\partial T = 0$ representing a given homology class and for which $M(T) = \min$; regularity theorems in dimension $n \leqq 7$ now imply that spt T is a smooth manifold. Lawson [L] proved that if $M = S^3$ then M contains closed minimal surfaces of arbitrarily high genus. For the general case, there has been a partially successful approach by Almgren [AF 2].

Let I^m be the unit m-cube, I_0^m its boundary, let (A, a) be a space with a base point a and let $\pi_m(A, \{a\})$ be the m dimensional homotopy group of equivalence classes of continuous mappings of (I^m, I_0^m) into (A, a). If $Z_k(M, G)$ is the group [1]) of k-cycles of M with coefficients in the abelian group G, then it is known that $\pi_m\big(Z_k(M, G), \{0\}\big)$ is naturally isomorphic with the homology group $H_{m+k}(M; G)$ for $1 \leqq k \leqq \dim M$. Now let Π be a homotopy class of maps $\varphi\colon (I^m, I_0^m) \to \big(Z_k(M, G), \{0\}\big)$ and consider the minimax problem

$$\inf_{\varphi \in \Pi} \sup_{t \in I^m} F(\varphi(t))$$

where F is a good function on $Z_k(M, G)$, in our case the mass. The point of Morse theory is: if $\Pi \neq 0$, then the solutions to the minimax problem are non-trivial critical points of the functional $F(T)$. Almgren succeeded in doing this on the space of k-currents and k-varifolds, obtaining non-trivial stationary varifolds in this way. Unfortunately, the regularity theory of stationary varifolds is still at a rudimentary stage, and Almgren's solution suffers of the same defects as for the earlier Federer and Fleming solution to Plateau's problem. However, Pitts has been able to restrict the class of competing maps φ in such a way so that the topological aspects are unchanged but the critical points share the most fundamental properties of locally minimizing currents and varifolds. For these special critical points he is then able to carry further the regularity theory and obtain eventually his theorem.

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[1]) Here $Z_k(M, G)$ is made into a topological group by means of the flat norm.